Arakelov motivic cohomology

Andreas Holmstrom

Clare College

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Abstract

The subject of this thesis is a construction of a new cohomology theory for arithmetic schemes, i.e. schemes flat and of finite type over $\operatorname{Spec}\mathbb{Z}$. We call this theory Arakelov motivic cohomology. It relates to motivic cohomology in the sense of Voevodsky roughly in the same way that the arithmetic Chow groups of Gillet and Soulé relate to ordinary Chow groups.

The motivation for constructing this cohomology theory comes from three sources. Firstly, a cohomology theory of this type plays a crucial role in Scholbach's conjecture on special values of L-functions, which can be seen as a reformulation of the Beilinson conjectures. Secondly, we hope that Arakelov motivic cohomology groups will eventually serve as a target for higher arithmetic Chern classes, and that it will be possible to develop higher arithmetic Riemann-Roch theory using these Chern classes. Finally, we have tried to make a construction which improves on earlier attempts to define similar groups, notably the work of Burgos and Feliu on higher arithmetic Chow groups.

Two of the key improvements compared to the groups of Burgos and Feliu is that Arakelov motivic cohomology groups are defined over arithmetic base schemes and not only over fields, and that they have some pushforward functoriality. The first is a precondition for any application to special values of L-functions, while the second is necessary for any formulation of higher arithmetic Riemann-Roch theory.

No result in this thesis depends on any unproven conjecture. This is large due to the fact that we can exploit the formalism of motivic stable homotopy theory, in particular the advances made in recent years by Ayoub, Cisinski and Déglise.

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration, except for the parts specified as collaborative work in the Introduction. No part of this material has ever been submitted for any other qualification.

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Statement of length

The lenght of this dissertation does not exceed the word limit set by the Degree Committee.

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Contents

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Chapter 1

Introduction

The aim of this thesis is to give a good construction of so called Arakelov motivic cohomology groups for arithmetic schemes, and begin to study their properties. Chapters 3, 4 and 5 contain joint work with Jakob Scholbach. Most of the key ideas in these chapters occurred independently to myself and Scholbach, and we are now writing up these things together in [34] and possibly other additional articles. I would like to emphasize our gratitude to Denis-Charles Cisinski and Frédéric Déglise for many crucial conversations during the process of developing these ideas.

Chapter 6 contains joint work in progress with Peter Arndt.

1.1 The idea of Arakelov motivic cohomology

1.1.1 Motivic cohomology

The idea of mixed motives, going back to Deligne and Beilinson (and maybe also Grothendieck) is that the category of mixed motives should form a universal cohomology theory for the category of all varieties, not necessarily smooth or projective. The models for such a theory come from etale cohomology and Hodge theory, which show many remarkable similarities indicating that they have a common origin. There are two ways to realize this idea - the most ambitious one would be to look for an abelian category of mixed motives, as described for example in [40]. Such a category has not yet really been constructed, although there are interesting suggestions by Nori and others. A slightly less ambitious goal would be to construct a triangulated category of mixed motives, originally thought of as the derived category of the abelian category. Such a triangulated category was constructed by Voevodsky ([66], [45]), and the theory has recently been generalized by Cisinski and Déglise [19] to base schemes which are not fields.

Assuming a theory of mixed motives over a base scheme S, one can define motivic cohomology for S-schemes either in terms of Ext groups in an abelian category of mixed motives, or as Hom groups in a triangulated category of mixed motives (see [40, sections 3 and 4]). The universality of mixed motives translates into the statement that motivic cohomology should be the universal Bloch-Ogus theory. The axioms for a Bloch-Ogus cohomology was first described in [7], and are presented in a slightly different form in [40].

There are two simplified settings in which motivic cohomology can be described using tools existing already before Voevodsky's work. Firstly, if we restrict attention to schemes which are smooth over a field, one can define motivic cohomology in terms of Bloch's higher Chow groups [5]. Secondly, if we accept working with rational coefficients, i.e. all cohomology groups tensored with $\mathbb{Q}$, we can define motivic cohomology of any regular scheme in terms of Adams eigenspaces of algebraic K-theory [19, Cor 13.2.14]. However, if we want to work in more general situations, we need to work with some category of mixed motives, or alternatively with motivic stable homotopy theory.

The idea of motivic stable homotopy theory, going back to Morel and Voevodsky, is that it should be possible to mimick the construction of the usual topological stable homotopy category **SH**. An object in this category is called a spectrum, and is essentially a sequence $\{K_n\}$ of spaces, together with maps $\Sigma K_n \to K_{n+1}$ from the suspension of the n-th space to the next. In topology, one of the functions of the stable homotopy category is that it receives a functor from the category of spaces, and in **SH** every cohomology theory becomes a representable functor. Conversely, every object in **SH** represents a cohomology theory on the category of spaces. Here a cohomology theory is by definition a family of functors satisfying the Eilenberg-Steenrod axioms. Following Morel, Voevodsky and Ayoub, one can construct a "motivic stable homotopy category" $\mathbf{SH}(S)$ for very general base schemes S, which plays a role in algebraic geometry similar to that of $\mathbf{SH}$ in topology, in that objects in $\mathbf{SH}(S)$ represent cohomology theories on the category of finite type S-schemes. A difference compared to the topological setting is that there is no precise axiomatic characterization of the cohomology theories which are representable by a motivic spectrum, but the main principle is that any cohomology theory which is $\mathbb{A}^1$-homotopy invariant and satisfies Nisnevich descent should be representable. The basic idea of the construction of $\mathbf{SH}(S)$ is that the category $\mathbf{Sm}/S$ of smooth S-schemes embeds into the category of simplicial presheaves on $\mathbf{Sm}/S$ via a Yoneda embedding, and the latter category behaves like the category of topological spaces, so one can copy the construction in topology and obtain a definition of $\mathbf{SH}(S)$. In [65], Voevodsky defines a spectrum which represents motivic cohomology. Working with rational coefficients, one can also define the motivic cohomology spectrum as a suitable Adams-graded piece of the K-theory spectrum, following Riou [51].

1.1.2 Arakelov motivic cohomology

The definitions of Chow groups, algebraic K-theory and motivic cohomology make sense for very general schemes, in particular one can define these theories for varieties over fields as well as for schemes of finite type over arithmetic bases like $\operatorname{Spec}\mathbb{Z}$. However, when working over $\operatorname{Spec}\mathbb{Z}$, one would like to also have Arakelov-theoretic versions of these theories, which take into account data at the infinite prime in some suitable sense, and this is the picture to which this thesis aims to make a contribution. Much work has been done to construct Arakelov-theoretic versions of Chow groups and K-theory, which are usually referred to as *arithmetic* Chow groups and *arithmetic* K-theory respectively.

The idea of adding data at infinity to Chow groups and K-theory seems to go back to work of Soulé, Gillet and Deligne in the 80s. The work of Gillet and Soulé on arithmetic Chow groups was presented in their classical article [30]. We review arithmetic Chow groups in section 2.4, but note for now that these groups are defined in particular for regular schemes which are flat and of finite type over $\operatorname{Spec}\mathbb{Z}$, as well as for smooth quasi-projective varieties over $\mathbb{Q}$. A different version of these groups (agreeing with the original one for projective schemes) was studied by Burgos in [10]. In the case of K-theory, there is a definition of arithmetic K_0 due to Gillet and Soulé [31] as well as several versions of higher arithmetic K-theory, due to Deligne, Soulé and Takeda. Recently Feliu and Burgos defined higher arithmetic Chow groups [11], but only for varieties over a number field and not for schemes of finite type over $\operatorname{Spec}\mathbb{Z}$ or other rings of integers. These groups have several nice functorial and structural properties, improving a lot on earlier work of Goncharov, the two main drawbacks being that the definition does not work over arithmetic base schemes, and that there is no pushforward functoriality for proper morphisms.

The original aim of this thesis project was to define a notion of higher arithmetic Chow groups which would improve on these two problems, while retaining all the good properties established for the Burgos-Feliu groups. The outcome is a theory which is more in the spirit of motivic cohomology than higher Chow groups, so we have called it Arakelov motivic cohomology (avoiding the term arithmetic motivic cohomology because usual motivic cohomology was at some point referred to as arithmetic cohomology).

The basic idea in any definition of higher arithmetic K-groups or higher arithmetic Chow groups is to define these groups as the homotopy fiber of some version of the Beilinson regulator. In order to do so, one must lift the Beilinson regulator from the level of cohomology groups to some category in which the notion of homotopy fiber (or cone) makes sense, for example to a map between complexes which compute the domain and target of the regulator. For example, Burgos and Feliu do this by exhibiting a zigzag of maps between explicit complexes calculating higher Chow groups and Deligne cohomology, respectively. The result of any definition in the spirit will be a long exact sequence relating the arithmetic theory, the usual theory, and Deligne cohomology. In the setting of this thesis, we obtain a lift of the regulator to the motivic stable homotopy category $\mathbf{SH}(\operatorname{Spec}\mathbb{Z})$, and as a result get a long exact sequence

$$\cdots \widehat{\mathrm{H}}^n(X,\mathbb{Z}(p)) \to \mathrm{H}^n_{\mathrm{M}}(X,\mathbb{Z}(p)) \to \mathrm{H}^n_{\mathrm{D}}(X,\mathbb{R}(p)) \to \widehat{\mathrm{H}}^{n+1}(X,\mathbb{Z}(p)) \cdots$$

for X a sufficiently nice arithmetic scheme (see Prop 3.4.1 and section 4.2). The items in this sequence are Arakelov motivic cohomology of X, motivic cohomology of X, and Deligne cohomology of the generic fiber of X.

There are two conceptual ways of thinking about Arakelov motivic cohomology, which as far as I understand should be viewed as guidelines for the intuition rather than precise mathematical statements. Firstly, the long exact sequence above can be thought of as a localization sequence for the inclusion of an arithmetic scheme into its Arakelov compactification. I believe this idea goes back to Beilinson. In this picture, motivic cohomology is the cohomology of the arithmetic scheme, Arakelov motivic cohomology is the cohomology of its compactification (i.e. the scheme with a fiber at infinity added), and Deligne cohomology is the cohomology of the fiber at infinity or maybe a punctured disc around this fiber.

Secondly, one can think of Arakelov motivic cohomology as *cohomology with compact support*, and indeed both Scholbach [55] and Flach [26] have formulated conjectures on the existence of Arakelov motivic cohomology using the notation H_c and $\mathrm{R}\Gamma_c$. A crucial point here is that for an arithmetic scheme, there are two distinct meanings of compact support. The traditional notion of motivic cohomology with compact support, as defined for example in section 2.2.3, should be thought of as compact support in the vertical direction, i.e. in the direction along the fibers. The other notion of compact support, which is the one we have in mind in this analogy, is in some sense compact support in the horizontal direction. One reason to think of Arakelov motivic cohomology in this way is that Scholbach has given a reformulation of the Beilinson conjectures which takes the form of a duality pairing between motivic cohomology and Arakelov motivic cohomology, whose rank and determinant should determine special values of L-functions and zeta functions. This "motivic global duality" has some formal similarities with global arithmetic duality as described in the book of Milne [46] (involving cohomology with compact support of a constructible etale sheaf on a ring of integers Spec $\mathcal{O}_F$), as well as with Poincare duality on a curve. In both the two latter settings some version of cohomology with compact support occurs as one of the factors in the duality pairing. The case of Poincare duality also fits well with the idea of Spec $\mathbb{Z}$ as analogous to an affine curve.

1.2 Further motivation and potential applications

To complete the Arakelov-theoretic picture of K-theory, Chow groups and motivic cohomology described above would have been sufficient motivation for the constructions in this thesis, but there is also additional motivation, coming firstly from the study of special values of L-functions and zeta functions, and secondly from arithmetic Riemann-Roch theory.

1.2.1 Special values of L-functions and zeta functions

There seem to have been a general expectation among experts that some parts of the above Arakelov-theoretic picture should be related to special values in some way. I first learnt about this from private conversations with Burgos, Soulé and Scholl, and also through remarks in survey articles by Flach and by Goncharov. Recently several attempts have been made to formulate precise conjectures on such a connection. So far, the most elaborate conjecture has been formulated by Scholbach in his thesis, see [56] and [55]. This gives a conjecture for special L-values (up to a rational factor) of motives over Spec $\mathbb{Z}$ (motives in the sense of Cisinski and Déglise). Under common (optimistic) assumptions on properties of algebraic cycles, existence of a motivic t-structure, and meromorphic continuation and functional equation of L-functions, the conjecture of Scholbach is equivalent to the conjunction of Beilinson's conjecture and the pole order conjectures of Soulé and Tate.

When it comes to predicting precise special values, i.e. avoiding the undetermined rational factor, there are two lines of thought which may eventually lead to a good connection with the Arakelov-theoretic picture above. Firstly, there is a recent letter of Soulé to Bloch and Lichtenbaum [62], in which the idea is that special values might be expressed as volumes of certain (conjecturally compact) Arakelov motivic cohomology groups. Secondly, there are some recent ideas of Flach, building on earlier work of Flach-Morin and Lichtenbaum, using the idea of Weil-etale topology. In [27], Flach and Morin give a conjecture and prove some result for Hasse-Weil zeta functions at the integer $s = 0$. It is expected that for other integers, some version of Arakelov motivic cohomology will play a role in some way, together with some form of the Weil-etale topology.

Since we learnt about the work of Flach and Morin only recently, the construction in this thesis has been guided mainly by the conjecture of Scholbach and to some extent by the letter of Soulé. For the framework of Flach, a more refined definition of Arakelov motivic cohomology will have to be given in the future. In chapter 5, we review the ideas of Scholbach and Soulé, and give some reason to believe that the groups we define are a reasonable candidate for the conjecture of Scholbach.

1.2.2 Arithmetic Riemann-Roch theory

A key fact in the setting of algebraic K-theory as well as arithmetic K_0 is that there is a Riemann-Roch square which explains precisely the failure of Chern classes to commute with push-forwards. A natural hope is that some version of higher arithmetic K-groups also fits into such a Riemann-Roch square, but until now there has been no suitable target cohomology for a theory of higher arithmetic Chern classes (with the exception of Burgos-Feliu arithmetic Chow theory, which lacks pushforward functoriality). We hope that Arakelov motivic cohomology groups will provide such a target, and that it will eventually be possible to prove a higher arithmetic Riemann-Roch theorem.

1.3 Outline

In chapter 2 we review some background material on motivic homotopy theory, Deligne cohomology, and arithmetic Chow groups.

In chapter 3 we construct a motivic spectrum over the base scheme $\mathrm{Spec}\,\mathbb{Z}$ which will represent Arakelov motivic cohomology. The first step is to construct a spectrum representing Deligne cohomology. This is done be generalizing a method used by Cisinski and Deglise in the setting of mixed Weil cohomologies. The second step is to apply a theorem of Cisinski and Deglise which produces a regulator map from the motivic cohomology spectrum to the Deligne cohomology spectrum. The Arakelov motivic cohomology spectrum is then defined as the homotopy fiber of this regulator map. A standard recipe using Ayoub's six functors formalism associates a cohomology theory for finite type S-schemes to any spectrum over the base S, and we apply this to define Arakelov motivic cohomology for schemes of finite type over $\mathrm{Spec}\,\mathbb{Z}$. Because this cohomology is defined by the formalism of motivic homotopy theory, it enjoys a number of good formal properties.

When working with schemes over a base field, the standard recipe just mentioned is the only reasonable way to construct a cohomology theory from a spectrum. However, when working over an arithmetic base, it turns out that there are two reasonable constructions, and in chapter 4 we point out this surprising phenomenon, and discuss some consequences of choosing the alternative definition of Arakelov motivic cohomology.

In chapter 5, we review some recent conjectures on zeta values and discuss how Arakelov motivic cohomology is expected to fit into these conjectural frameworks.

In chapter 6, we show that the Arakelov motivic cohomology spectrum can be equipped with the structure of a non-unital ring spectrum, thereby giving a notion of products on the Arakelov motivic cohomology groups.

In chapter 7, which is independent from the rest of the thesis, we discuss a gluing formalism for spectra over $\mathrm{Spec}\,\mathbb{Z}$.

1.4 Work in progress and future directions

In the months preceding the submission of this thesis, I have (together with Jakob Scholbach) obtained some results which for time reasons I was not able to include in this thesis. The best I can do now is to give an indication of what these results are, and refer to [34] and other forthcoming preprints for the proofs and precise statements.

The first main point is that we can prove that the regulator map we define agrees with the Beilinson regulator on the level of cohomology groups. This is not surprising, given the canonical nature of our definition, but it still requires some work. The proof uses a three-step method familiar from other similar comparison theorems in the literature: First one reduces the regulator on all algebraic K-groups to the case of K_0 (using techniques from Riou's thesis), and then one uses the splitting principle to reduce to the case of the Picard group, on which one can prove agreement directly.

Secondly, we believe that we can obtain comparison results with higher arithmetic Chow groups in the sense of Burgos and Feliu (for varieties over $\mathbb{Q}$), as well as with arithmetic Chow groups in the sense of Gillet and Soulé (for arithmetic schemes). For the Burgos-Feliu groups we expect agreement for all smooth varieties, modulo the fact that Burgos and Feliu choose to work with a truncated version of Deligne cohomology, while we prefer to work with the non-truncated version, for reasons related to duality and the expected functional equation for motivic L-functions. In terms of Hodge cohomology, this is essentially the same as saying that we prefer to forget about the weight filtration. For the Gillet-Soulé groups, we expect that for a sufficiently nice arithmetic scheme X, $\widehat{\mathrm{H}}^{2p}(X,p)$ should agree with $\overline{\mathrm{CH}}^p(X)$, the subgroup of $\widehat{\mathrm{CH}}^p(X)$ corresponding to homologically trivial cycles (see section 2.4.1). In this setting we also hope to eventually give a new definition and a generalization of the height pairing, using that the spectrum representing Arakelov motivic cohomology with rational coefficients is a module over the motivic cohomology spectrum.

In the final chapter, we list a number of other interesting open questions and directions for future research.

Chapter 2

Preliminaries

2.1 Conventions and terminology

For simplicity we work throughout with schemes of finite type over $\operatorname{Spec} \mathbb{Z}$ and varieties over $\mathbb{Q}$ rather than over a more general rings of integers $\operatorname{Spec} \mathcal{O}_F$ and number fields, although there should be no essential difficulty in doing everything in the more general setting.

All schemes are assumed to be Noetherian and separated. By "arithmetic scheme", we shall mean a scheme flat and of finite type over $\operatorname{Spec} \mathbb{Z}$ (not necessarily regular or projective).

When we say Deligne cohomology, we will always mean *real* Deligne cohomology with real coefficients, i.e. we will always take invariants under infinite Frobenius. This is usually written $\mathrm{H}^n_{\mathrm{D}}(V_{/\mathbb{R}}, \mathbb{R}(p))$, but we will write $\mathrm{H}^n_{\mathrm{D}}(V, \mathbb{R}(p))$ or even $\mathrm{H}^n_{\mathrm{D}}(V, p)$.

2.2 Motivic homotopy theory

In this section we review some background material from motivic homotopy theory. Most of this material is taken from the two papers [20] and [19] of Cisinski and Déglise.

2.2.1 Various motivic categories

Let S be a Noetherian scheme of finite Krull dimension. The aim of this section is to review the definitions of the following categories associated to S.

- The motivic stable homotopy category $\mathbf{SH}(S)$.
- For a ring Λ, a category $\mathbf{D}_{\mathbb{A}^1,\Lambda}(S)$. This can be thought of as a Λ-linear version of $\mathbf{SH}(S)$.
- The category $\mathbf{DM}_{\mathrm{B}}(S)$ of *Beilinson motives* over S. Roughly speaking, this is the subcategory of $\mathbf{D}_{\mathbb{A}^1,\mathbb{Q}}$ consisting of modules over the *Beilinson spectrum* H_{B} (defined below). If $S = \operatorname{Spec} k$, with k a perfect field, this category is equivalent to Voevodsky's triangulated category of motives (with rational coefficients), which is usually denoted $\mathbf{DM}_{\mathbb{Q}}(k)$.
- The category $\mathbf{DM}_{\mathrm{gm}}(S)$. This is the full subcategory of compact objects in $\mathbf{DM}_{\mathrm{B}}$, and following Voevodsky we refer to this as the subcategory of geometric motives.

Let $\mathbf{Sm}/S$ be the category of finite type, smooth S-schemes. For any ring Λ, we write $\mathbf{Shv}(\mathbf{Sm}/S, \Lambda)$ for the category of Nisnevich sheaves of Λ-modules on $\mathbf{Sm}/S$. We also write $\mathbf{Comp}(\mathbf{Sm}/S, \Lambda)$ for the category of complexes of such sheaves, and $\mathbf{D}(\mathbf{Sm}/S, \Lambda)$ for the associated (unbounded) derived category.

Any object X of $\mathbf{Sm}/S$ represents a presheaf of sets on $\mathbf{Sm}/S$. We can compose this presheaf with the free Λ-module functor, and then sheafify for the Nisnevich topology. This gives an object of $\mathbf{Shv}(\mathbf{Sm}/S, \Lambda)$ which we denote by $\Lambda(X)$.

Definition 2.2.1. Let $\mathbf{L}$ be the localizing subcategory of the triangulated category $\mathbf{D}(\mathbf{Sm}/S, \Lambda)$ generated by complexes of the form

$$\ldots \to 0 \to \Lambda(X \times_S \mathbb{A}^1_S) \to \Lambda(X) \to 0 \to \ldots$$

where the middle map is induced by projection. We define the category $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(S)$ to be the quotient $\mathbf{D}(\mathbf{Sm}/S, \Lambda)/\mathbf{L}$. We say that a map in $\mathbf{Comp}(\mathbf{Sm}/S, \Lambda)$ is an $\mathbb{A}^1$-equivalence if its image in $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}$ is an isomorphism.

Definition 2.2.2. Consider a complex K of presheaves of Λ-modules on $\mathbf{Sm}/S$. We say that K is $\mathbb{A}^1$-homotopy invariant if for every X in $\mathbf{Sm}/S$, the map $K(X) \to K(X \times_S \mathbb{A}^1_S)$, induced by projection, is a quasi-isomorphism. We say that K is $\mathbb{A}^1$-local if the map $\mathbb{H}^n(X, K_{\mathrm{Nis}}) \to \mathbb{H}^n(X \times_S \mathbb{A}^1_S, K_{\mathrm{Nis}})$ is an isomorphism for all X and n. Here $\mathbb{H}$ denotes Nisnevich hypercohomology and K_{Nis} means the Nisnevich sheafification of K. Finally, we say that K is Nisnevich local if the canonical map $H^n(K(X)) \to \mathbb{H}^n(X, K_{Nis})$ is an isomorphism for all X and n. (This last map is the map induced by the Nisnevich sheafification functor, from a Hom group in the derived category of presheaves, to a Hom group in the derived category of sheaves.)

Remark 2.2.3. A few words about terminology: The property of being Nisnevich local is also referred to as satisfying Nisnevich descent. It is equivalent to the Brown-Gersten property, and also to the Nisnevich excision property. See [20, Prop 1.1.10] for the definitions of these properties and the proof of their equivalence.

Proposition 2.2.4. *[20, Prop 1.1.15] There is a model structure on the category* $\mathbf{Comp}(\mathbf{Sm}/S, \Lambda)$ *such that the associated homotopy category is* $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(S)$. *In this model structure the weak equivalences are precisely the* $\mathbb{A}^1$*-equivalences, and an object is fibrant if and only if it is* $\mathbb{A}^1$*-homotopy invariant and Nisnevich local.*

Proposition 2.2.5. *[20, Cor 1.1.17] The localization functor*

$$\mathbf{D}(\mathbf{Sm}/S, \Lambda) \to \mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}$$

has a right adjoint which is fully faithful, with essential image the $\mathbb{A}^1$*-local complexes.*

Definition 2.2.6. Write Λ for the object $\Lambda(S)$ represented by the base scheme. We define the Tate object $\Lambda(1)$ as $\operatorname{coker}(u)[-1]$, where $u : \Lambda \to$

$\Lambda(\mathbb{G}_m)$ is the map induced by the unit of $\mathbb{G}_m$. This definition implies in particular that $\Lambda(\mathbb{G}_m) \cong \Lambda \oplus \Lambda(1)[1]$. For any object M, we define the Tate twist $M(1)$ of M to be $M \otimes \Lambda(1)$.

Next we recall the idea of symmetric spectra, which we will employ in order to invert the Tate object. Let $\mathcal{C}$ be a symmetric monoidal category and let T be an object of $\mathcal{C}$. A symmetric T-spectrum in $\mathcal{C}$ is a sequence of objects $\{F_n\}_{n=0}^{\infty}$ together with actions of the symmetric group S_n on F_n and "bonding" morphisms $T \otimes F_n \to F_{n+1}$ such that the induced maps $T^{\otimes m} \otimes F_n \to F_{n+m}$ are $S_m \times S_n$-equivariant. A morphism of symmetric spectra from $F = \{F_n\}_{n=0}^{\infty}$ to $F' = \{F'_n\}_{n=0}^{\infty}$ is a sequence of morphisms $a_n : F_n \to F'_n$ which are S_n-equivariant and commute with the bonding maps.

Definition 2.2.7. A symmetric Tate spectrum is a symmetric T-spectrum in the category $\mathbf{Comp}(\mathbf{Sm}/S, \Lambda)$, where we take T to be the Tate object $\Lambda(1)$. We write $\mathbf{Sp}_{Tate}(S)$ for the category of symmetric Tate spectra.

There is an adjunction

$$\Sigma^{\infty} : \mathbf{Comp}(\mathbf{Sm}/S, \Lambda) \rightleftarrows \mathbf{Sp}_{Tate}(S) : \Omega^{\infty}$$

These functors are defined by

$$\Omega^{\infty} : \{F_n\} \mapsto F_0$$

and

$$\Sigma^{\infty} : A \mapsto \{A(n)\}_{n=0}^{\infty}$$

with each bonding map being the identity map, and group action on $A(n) = \Lambda(1)^{\otimes n} \otimes A$ given by permuting the factors in $\Lambda(1)^{\otimes n}$.

There is a closed symmetric monoidal structure on $\mathbf{Sp}_{Tate}(S)$, uniquely determined by the requirements that Σ^{∞} is a symmetric monoidal functor and that $\Sigma^{\infty}\Lambda(S)$ is the unit for the tensor product.

Definition 2.2.8. We say that a map $F \to F'$ of symmetric Tate spectra is a quasi-isomorphism if $F_n \to F'_n$ is a quasi-isomorphism of complexes for each

n. We define the Tate derived category $\mathbf{D}_{Tate}(S, \Lambda)$ to be the localization of $\mathbf{Sp}_{Tate}$ by quasi-isomorphisms.

A symmetric Tate spectrum F is said to be a weak Ω-spectrum if for all n, the transpose $F_n \to \mathbf{RHom}(\Lambda(1), F_{n+1})$ of the bonding map is an isomorphism in $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(S, \Lambda)$. We say that F is an Ω-spectrum if in addition each F_n is Nisnevich local and $\mathbb{A}^1$-homotopy invariant.

A morphism $u : A \to B$ of symmetric Tate spectra is a stable $\mathbb{A}^1$-equivalence if for any weak Ω-spectrum F, the map $u^* : \mathrm{Hom}_{\mathbf{D}_{Tate}}(B, F) \to \mathrm{Hom}_{\mathbf{D}_{Tate}}(A, F)$ is an isomorphism.

Proposition 2.2.9. *[20, Prop 1.4.3] There is a stable symmetric monoidal model structure on the category of symmetric Tate spectra in which the weak equivalences are the stable $\mathbb{A}^1$-equivalences. In this model structure, tensoring with any cofibrant object preserves weak equivalences. The functor Σ^∞ is a symmetric monoidal left Quillen functor, sending $\mathbb{A}^1$-equivalences to stable $\mathbb{A}^1$-equivalences.*

Definition 2.2.10. We define the category $\mathrm{D}_{\mathbb{A}^1,\Lambda}(S)$ as the homotopy category of symmetric Tate spectra with respect to the above model structure.

Proposition 2.2.11. *[20, Prop 1.4.7] Let $E = \{E_p\}_{p=0}^{\infty}$ be a weak Ω-spectrum. Then for any integer $n \geq 0$ and any smooth, finite type S-scheme X, there are isomorphisms*

$$\mathrm{H}^n(E_p(X)) \cong \mathrm{Hom}_{\mathrm{D}_{\mathbb{A}^1,\Lambda}(S)}(\Lambda(X), E(p)[n])$$

The definition of the motivic stable homotopy category $\mathbf{SH}(S)$ is completely analogous to the definition of $\mathrm{D}_{\mathbb{A}^1,\Lambda}$, when we replace complexes of sheaves of Λ-modules by simplicial sheaves. More precisely, let $\Delta^{\mathrm{op}}\mathbf{Shv}(\mathbf{Sm}/S)$ be the category of Nisnevich sheaves of simplicial sets on $\mathbf{Sm}/S$, and write $\Delta^{\mathrm{op}}\mathbf{Shv}_\bullet(\mathbf{Sm}/S)$ for the pointed version of this category. Let $T \in \Delta^{\mathrm{op}}\mathbf{Shv}_\bullet(\mathbf{Sm}/S)$ be the quotient sheaf $\mathbb{A}^1/\mathbb{G}_m$, and let $\mathbf{Sp}(S)$ be the category of symmetric T-spectra, also referred to as motivic symmetric spectra. The motivic stable homotopy category $\mathbf{SH}(S)$ is the homotopy category of $\mathbf{Sp}(S)$ with respect to stable $\mathbb{A}^1$-equivalences. Here stable $\mathbb{A}^1$-equivalences are defined just as above, replacing the effective category $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}$ by the unstable motivic homotopy category $\mathcal{H}(S)$ and the Tate derived category $\mathbf{D}_{Tate}$ by the category of motivic symmetric spectra localized at levelwise weak equivalences. See [37] for more details. We also introduce the notation $\mathbf{SH}_{\mathbb{Q}}(S)$ for the rationalization of $\mathbf{SH}(S)$, i.e. the category with the same objects but all Hom groups tensored with $\mathbb{Q}$. ($\mathbf{SH}_{\mathbb{Q}}(S)$ can also be expressed as the Verdier quotient of $\mathbf{SH}(S)$ by the localizing subcategory generated by compact torsion objects, see [19, 5.3.37]).

Definition 2.2.12. Let $\mathbf{Sp} = \mathbf{Sp}(S)$ be either the category of motivic symmetric spectra or the category of symmetric Tate spectra over S, and let $\mathbf{Ho}(\mathbf{Sp})$ be the associated homotopy category, i.e. $\mathbf{SH}(S)$ or $\mathrm{D}_{\mathbb{A}^1,\Lambda}(S)$ respectively. A weak ring spectrum is a commutative monoid object in $\mathbf{Ho}(\mathbf{Sp})$. A strict ring spectrum is a weak ring spectrum E such that there exists a commutative monoid object E' in $\mathbf{Sp}$ and an isomorphism of weak ring spectra between E and E'.

Now we turn to Cisinski and Déglise's category of Beilinson motives. Recall that S is assumed to be a Noetherian scheme of finite Krull dimension. Let $KGL_{\mathbb{Q},S} \in \mathbf{SH}(S)$ be the spectrum representing algebraic K-theory with rational coefficients [17], [49]. Riou [51] showed that there is a decomposition

$$KGL_{\mathbb{Q},S} \simeq \oplus_{i\in\mathbb{Z}} KGL_S^{(i)}$$

compatible with base change and such the summands represent the Adams-graded pieces of algebraic K-theory in the sense that if S is regular, there is an isomorphism

$$\mathrm{Hom}_{\mathrm{D}_{\mathbb{A}^1,\mathbb{Q}}(S)}(\mathbb{Q}[n], KGL_S^{(i)}) \simeq K_n^{(i)}(S) := Gr^i_\lambda K_n(S)_{\mathbb{Q}}.$$

Definition 2.2.13. The Beilinson spectrum over S is defined as $\mathrm{H}_{\mathrm{B}} = \mathrm{H}_{\mathrm{B},S} := KGL_S^{(0)}$.

Definition 2.2.14. By [19, Cor 13.2.6], the Beilinson spectrum is a strict ring spectrum, so we can consider the category of modules over it inside the category of symmetric Tate spectra with rational coefficients. This module category can be equipped with a model structure in which weak equivalences are those maps which are weak equivalences in the underlying category of spectra. We define the category of Beilinson motives over S, denoted $\mathbf{DM}_{\text{B}}(S)$, to be the homotopy category of this module category: $\mathbf{DM}_{\text{B}}(S) := \mathbf{Ho}(\text{H}_{\text{B}} - mod)$.

Definition 2.2.15. We define the triangulated category $\mathbf{DM}_{\text{gm}}(S)$ of geometric motives over S to be the full subcategory of compact objects in $\mathbf{DM}_{\text{B}}(S)$.

For more details on Beilinson motives and geometric motives, see [19, Chapters 13, 14, 15].

2.2.2 Six functors formalism

By work of Ayoub [2], [3], the motivic stable homotopy category **SH** satisfies a so called six functors formalism. Cisinski and Déglise [19] have showed that the same is true for $\mathbf{DM}_{\text{B}}$ as defined above. In the next four theorems we summarize some basic properties of this formalism. Here all schemes are assumed to be Noetherian and of finite Krull dimension.

Theorem 2.2.16. *Let* $\mathbf{T}$ *denote either* **SH** *or* $\mathbf{DM}_{\text{B}}$*. Then the following properties hold.*

- *For any scheme X, the category $\mathbf{T}(X)$ is closed symmetric monoidal.*
- *For any morphism $f : Y \to X$, there is a pair of adjoint functors*

$$f^* : \mathbf{T}(X) \rightleftarrows \mathbf{T}(Y) : f_*$$

  and f^* is a monoidal functor.
- *For any separated morphism of finite type $f : Y \to X$, there is a pair of adjoint functors*

$$f_! : \mathbf{T}(Y) \rightleftarrows \mathbf{T}(X) : f^!$$

- *For any morphism f separated of finite type, there is a natural transformation $f_! \to f_*$ which is an isomorphism if f is proper.*

- *If f is an open immersion, we have $f^! = f^*$.*

- *For any cartesian square*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array}$$

  where f is separated of finite type, there are natural isomorphisms of functors

$$g^* f_! \xrightarrow{\sim} f'_! g'^* \qquad \text{and} \qquad g'_* f'^! \xrightarrow{\sim} f^! g_*$$

  referred to as base change isomorphisms.

Theorem 2.2.17. *Let $\mathbf{T}$ denote either $\mathbf{SH}$ or $\mathbf{DM}_{\text{Б}}$. Let $Z \to X$ be a closed immersion with complementary open immersion $j : U \to X$. Then the following holds:*

- *The functor $j_!$ is left adjoint to j^*.*

- *The functor i_* is left adjoint to $i^!$.*

- *The functors $j_!$ and i_* are fully faithful.*

- *We have $i^* j_! = 0$*

- *For any object M of $\mathbf{T}$, there are natural distinguished triangles*

$$j_! j^! M \to M \to i_* i^* M \to j_! j^! M[1]$$

  and

$$i_! i^! M \to M \to j_* j^* M \to i_! i^! M[1]$$

  where the maps are given by units and counits of the relevant adjoint pairs of functors. These triangles are referred to as localization triangles.

Theorem 2.2.18 (Relative purity)**.** *Consider now the case of* $\mathbf{DM}_{\text{B}}$ *only. Let* $f : Y \to X$ *be a smooth quasi-projective morphism of relative dimension* d*. Then for any* M *in* $\mathbf{DM}_{\text{B}}(X)$*, there is a natural isomorphism* $f^*(M)(d)[2d] \simeq f^!(M)$

When talking about the category $\mathbf{DM}_{\text{B}}(X)$ in the next theorem and in the future, we will sometimes write $\mathbf{1}_X$ instead of $\mathrm{H}_{\text{B},X}$, motivated by the fact that this is the unit object of the symmetric monoidal structure on $\mathbf{DM}_{\text{B}}(X)$.

Theorem 2.2.19 (Absolute purity)**.** *For any closed immersion between regular schemes* $i : Z \to X$ *of codimension* c*, we have a canonical isomorphism* $\mathbf{1}_Z(-c)[-2c] \simeq i^!\mathbf{1}_X$ *in* $\mathbf{DM}_{\text{B}}(Z)$

2.2.3 Cohomology theories

Let $\mathbf{T}$ denote either $\mathbf{SH}$ or $\mathbf{DM}_{\text{B}}$, let $\mathbf{1}$ be the unit object for the tensor structure on $\mathbf{T}$, and consider an object E in $\mathbf{T}(S)$. For any S-scheme $f : X \to S$ of finite type, we define E-cohomology of X (or cohomology with coefficients in E) by the formula

$$E^n(X,p) = \mathrm{Hom}_{\mathbf{T}(S)}(\mathbf{1}, f_* f^* E(p)[n]).$$

This defines a bigraded cohomology theory on the category of finite type S-schemes.

Although we will not use it in this thesis, we remark that one can also define the following theories associated to E:

- Homology: $E_n(X,p) = \mathrm{Hom}(\mathbf{1}, f_! f^! E(-p)[-n])$
- Cohomology with compact support: $E^n_c(X,p) = \mathrm{Hom}(\mathbf{1}, f_! f^* E(p)[n])$
- Borel-Moore homology: $E^{BM}_n(X,p) = \mathrm{Hom}(\mathbf{1}, f_* f^! E(-p)[-n])$

For us, an important example of a spectrum will be Voevodsky's motivic cohomology spectrum over a base scheme S, defined in ??, which we will denote by $\mathrm{H}\mathbb{Z}_S$. For X a finite type S-scheme, a naive application of the above general formula would give the following definition of motivic cohomology:

$$\mathrm{H}^n_{\mathrm{M}}(X,p) := \mathrm{Hom}_{\mathbf{SH}(S)}(\mathbf{1}, f_*f^*\mathrm{H}\mathbb{Z}_S(p)[n]) = \mathrm{Hom}_{\mathbf{SH}(X)}(\mathbf{1}, f^*\mathrm{H}\mathbb{Z}_S(p)[n]).$$

(Here the last equality comes from the adjointness of f^* and f_*, together with the fact that f^* a monoidal functor and hence preserves the unit object.) However, the usual definition of motivic cohomology is the following:

$$\mathrm{H}^n_{\mathrm{M}}(X,p) := \mathrm{Hom}_{\mathbf{SH}(X)}(\mathbf{1}, \mathrm{H}\mathbb{Z}_X(p)[n]).$$

Voevodsky conjectures [67, Conj 17] that $\mathrm{H}\mathbb{Z}_X \simeq f^*\mathrm{H}\mathbb{Z}_S$ in $\mathbf{SH}(X)$ for any morphism $f : X \to S$, and this would of course imply that the two definitions are equivalent. However, since this conjecture is not yet proven, we will distinguish between the two by referring to the first definition as *naive* motivic cohomology, and the second as *usual* motivic cohomology. A crucial point for us will be that Voevodsky's conjecture is true with rational coefficients, by a recent result of Cisinski and Déglise.

Theorem 2.2.20. *[19, Cor 15.1.6] Write* $\mathrm{H}\mathbb{Q} = \mathrm{H}\mathbb{Z} \otimes \mathbb{Q}$ *for Voevodsky's motivic cohomology spectrum with rational coefficients. For any excellent and geometrically unibranch scheme X, we have an isomorphism in* $\mathbf{SH}(X)$ *between the Beilinson spectrum* $\mathrm{H}_{\mathrm{B},X}$ *and* $\mathrm{H}\mathbb{Q}_X$*. Because $f^*H_{\mathrm{B},S}$ is always isomorphic to* $\mathrm{H}_{\mathrm{B},X}$*, this implies in particular that* $f^*\mathrm{H}\mathbb{Q}_S \simeq \mathrm{H}\mathbb{Q}_X$ *for any morphism $f : X \to S$ between excellent and geometrically unibranch schemes.*

For our purposes, this result means that whenever we work with rational or real coefficients, both definitions are equivalent, while if we work with integral coefficients, we have to be careful to specify which version of motivic cohomology we are talking about.

We end this section with the following theorem.

Theorem 2.2.21. *For any scheme S, there are functors*

$$\mathbf{DM}_{\mathrm{B}}(S) \to \mathrm{D}_{\mathbb{A}^1,\mathbb{Q}}(S) \to \mathbf{SH}_{\mathbb{Q}}(S)$$

the first being fully faithful, and the second being an equivalence.

Proof. The first functor can be expressed as a right adjoint of a Bousfield localization functor. It exists and is fully faithful by [19, Prop 13.2.3]. The second is a functor induced by the Eilenberg-Mac Lane functor of topology, see [3, Chapter 4], and is an equivalence according to [19, 5.3.37]. □

We note in particular the following immediate consequence.

Corollary 2.2.22. *Let E be an object of $\mathbf{DM}_{\mathrm{B}}(S)$. The cohomology theory associated to E is independent of whether we define it in $\mathbf{DM}_{\mathrm{B}}$, in $\mathrm{D}_{\mathbb{A}^1,\mathbb{Q}}$, or in $\mathbf{SH}$, i.e.*

$$\begin{aligned}\mathrm{Hom}_{\mathbf{DM}_{\mathrm{B}}(S)}(\mathrm{H}_{\mathrm{B}}, f_*f^*E(p)[n]) &\cong \mathrm{Hom}_{\mathrm{D}_{\mathbb{A}^1,\mathbb{Q}}(S)}(\mathbf{1}_{\mathbb{Q}}, f_*f^*E(p)[n]) \\ &\cong \mathrm{Hom}_{\mathbf{SH}(S)}(\mathbf{1}, f_*f^*E(p)[n])\end{aligned}$$

for any $f : X \to S$ of finite type, where $\mathbf{1}$ denotes the unit object of $\mathbf{SH}(S)$ and $\mathbf{1}_{\mathbb{Q}}$ denotes the unit object of $\mathrm{D}_{\mathbb{A}^1,\mathbb{Q}}(S)$.

Proof. The first isomorphism follows from the functor $\mathbf{DM}_{\mathrm{B}} \to \mathrm{D}_{\mathbb{A}^1,\mathbb{Q}}$ being fully faithful and the fact that the functor $(-) \otimes \mathrm{H}_{\mathrm{B}}$ is Quillen left adjoint to the forgetful functor from H_{B}-modules to rational spectra. The second follows from the equivalence between $\mathrm{D}_{\mathbb{A}^1,\mathbb{Q}}$ and the rationalization of $\mathbf{SH}$. □

2.3 Deligne cohomology

In this section we review the definiton and some basic properties of Deligne cohomology[1]. In the next chapter we use the Deligne complexes defined here to construct a spectrum in the sense of motivic stable homotopy theory, which represents Deligne cohomology for smooth varieties over k, where

[1] In some sources, the theory reviewed in this thesis is called Deligne-Beilinson cohomology, and the term Deligne cohomology is reserved for the cohomology obtained by dropping the “logarithmic at infinity” condition on the differential forms in the sheaves described below. This notion of Deligne cohomology is well-behaved only for smooth varieties which are also proper, and in this case it agrees with the theory reviewed here.

k is any subfield of $\mathbb{R}$. There are several closely related versions of Deligne cohomology and Hodge cohomology, and it is possible that some other version will be better suited for certain applications. The choice we make here is that of non-truncated Deligne cohomology (or weak Hodge cohomology), motivated by the conjecture of Scholbach.

2.3.1 Definition

Let k be a subfield of $\mathbb{R}$, and let V be a smooth variety over k. Let $j : V \hookrightarrow \overline{V}$ be an open immersion into a smooth proper variety, with the complement $D = \overline{V} \backslash V$ a normal crossings divisor. Write Ω^*_V for the complex of sheaves of holomorphic differential forms on $V(\mathbb{C})$ and let $\Omega^*_V(\log D)$ be the sheaf of forms with logarithmic singularities at infinity [21]. The stupid filtration $\sigma_{\geq *}$ of the complex $\Omega^*_V(\log D)$ is denoted F^* and called *Hodge filtration*. For any $p \in \mathbb{Z}$, $\mathbb{R}(p)$ denotes (the constant sheaf associated to) $(2\pi i)^p\mathbb{R} \subset \mathbb{C}$.

Definition 2.3.1. Let the *Deligne complex* be

$$\mathbb{R}_{\mathrm{D}}(p) := \mathrm{cone}(\mathrm{R}j_*\mathbb{R}(p) \oplus F^p\Omega^*_{\overline{V}}(\log D) \to \mathrm{R}j_*\Omega^*_V)[-1]. \tag{2.1}$$

The map of the first summand is induced by the standard map of complexes

$$\mathbb{R}(p) \subset \mathbb{C} \to \Omega^* = [\mathcal{O} \to \Omega^1 \to ...].$$

The second summand map comes from the adjunction, using that $j^*\Omega^*_{\overline{V}}(\log D) = \Omega^*_V$.

Deligne cohomology of V is defined by

$$\mathrm{H}^n_{\mathrm{D}}(V, p) := \mathbb{H}^n(\overline{V}(\mathbb{C}), \mathbb{R}_{\mathrm{D}}(p))^{\overline{F}_\infty}$$

where the right hand side denotes sheaf hypercohomology on $\overline{V}(\mathbb{C})$ and the superscript $\overline{F}_\infty$ indicates the subspace of $\overline{F}^*_\infty$-invariant elements. Here $\overline{F}_\infty$ is the so called infinite Frobenius, which acts by complex conjugation both on $\overline{V}(\mathbb{C})$ and on the Deligne complex.

This definition is independent of the choice of compactification $j : V \hookrightarrow$

$\overline{V}$, see [24, Lemma 2.8.].

2.3.2 Basic properties

We recall some basic properties of Deligne cohomology.

- There is a long exact sequence

$$\begin{aligned} \cdots \quad \to \quad & \mathrm{H}^n_{\mathrm{D}}(V,p) \to \mathrm{H}^n(V(\mathbb{C}),\mathbb{R}(p))^{(-1)^p} \to (\mathrm{H}^n_{\mathrm{dR}}(V_{\mathbb{R}})/F^p\mathrm{H}^n_{\mathrm{dR}}(V_{\mathbb{R}})) \\ \to \quad & \mathrm{H}^{n+1}_{\mathrm{D}}(V,p) \to \cdots . \end{aligned}$$

  Here F^p denotes the Hodge filtration. The second group denotes Betti cohomology, and the superscript $(-1)^p$ denotes the $(-1)^p$-eigenspace under the action of $\overline{F}_\infty$, i.e., the action induced by complex conjugation both on $V(\mathbb{C})$ and on the coefficients. This is a consequence of the definition and the degeneration of the Hodge to de Rham spectral sequence. See e.g. [24, Cor. 2.10].

- Deligne cohomology has Chern classes, in particular it receives a map from the Picard group, the *first Chern class*, see [13, Section 5.1.], and also [24, Section 7]:

$$c_1 : \mathrm{Pic}(V) \to \mathrm{H}^2_{\mathrm{D}}(V,1). \tag{2.2}$$

  In fact, there is a unique such morphism such that its composition with the map to Betti cohomology coincides with the usual first Chern class [24, Prop. 8.2.]. The first Chern class is a group homomorphism compatible with pullbacks.

- The Deligne complexes admit a product structure

$$\cup_\alpha : \mathbb{R}_{\mathrm{D}}(p)\otimes\mathbb{R}_{\mathrm{D}}(q) \to \mathbb{R}_{\mathrm{D}}(p+q) \tag{2.3}$$

  which depends on a parameter $\alpha \in [0,1]$, see [24, Section 3] for the definition and more details. This product is strictly associative for $\alpha = 0$ and $\alpha = 1$, and graded commutative for $\alpha = \frac{1}{2}$. The induced product on $\oplus_{p\geq 0}\mathrm{H}^*_{\mathrm{D}}(V,p)$ is independent of the choice of α, and hence it is commutative and associative. By "strictly" commutative or associative we mean commutative or associative in the category of complexes rather than in the homotopy category or derived category.

- Deligne cohomology is homotopy invariant: for any smooth variety V, the natural projection map $V\times\mathbb{A}^1 \to V$ induces an isomorphism

  $$\mathrm{H}^n_{\mathrm{D}}(V,p) \overset{\cong}{\to} \mathrm{H}^n_{\mathrm{D}}(V\times\mathbb{A}^1,p).$$

  This follows from the long exact sequence above and the homotopy invariance of Betti and de Rham cohomology.

- Deligne cohomology satisfies the projective bundle formula: let E be a vector bundle of rank r over a smooth variety V. Let $P := \mathbf{P}(E)$ be the projectivization of E. Let $\mathcal{O}_P(-1)$ be the tautological line bundle on P. Then there is an isomorphism

  $$\oplus_{i=0}^{r-1}\mathrm{H}^{n-2i}_{\mathrm{D}}(V,p-i) \overset{-\cup(c_1(\mathcal{O}_P(-1)))^i}{\longrightarrow} \mathrm{H}^n_{\mathrm{D}}(P,p).$$

  In particular the following "weak Künneth formula" holds:

  $$\mathrm{H}^n_{\mathrm{D}}(\mathbb{P}^1\times V,p) \cong \mathrm{H}^{n-2}_{\mathrm{D}}(V,p-1) \oplus \mathrm{H}^n_{\mathrm{D}}(V,p).$$

  See [24, Prop. 8.5.].

2.3.3 The Burgos complexes

In the next chapter, we construct a spectrum representing Deligne cohomology. As input to this construction, we need functorial complexes with products, which calculate Deligne cohomology (as opposed to complexes of sheaves whose hypercohomology gives Deligne cohomology). For this purpose, we review a construction of Burgos [10]. For the applications in this thesis, we only need to know Theorem 2.3.4 below, and not the precise details of Burgos' construction.

Definition 2.3.2. (Burgos) A *Dolbeault algebra* A is a differential graded $\mathbb{R}$-algebra $A_{\mathbb{R}}$, together with a "Hodge" decomposition of $A_{\mathbb{C}} := A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$

$$A^n_{\mathbb{C}} := \oplus_{p+q=n} A^{p,q},$$

such that on the complexified space d splits into $d = \partial + \overline{\partial}$, the two summands having degree $(1,0)$ and $(0,1)$, respectively, and such that the complex conjugation on $A_{\mathbb{C}}$ induces an isomorphism between $A^{p,q}$ and $A^{q,p}$.

For any Dolbeault algebra A, we write

$$F^p A := F^p A_{\mathbb{C}} := \oplus_{p',q;\ p' \geq p} A^{p',q}$$

for the Hodge filtration on $A_{\mathbb{C}}$, and $A_{\mathbb{R}}(p) := (2\pi i)^p A_{\mathbb{R}} \subset A_{\mathbb{C}}$.

Again for any Dolbeault algebra A, we put

$$\mathrm{D}^*(A, p) := \mathrm{cone}(A_{\mathbb{R}}(p) \oplus F^p A_{\mathbb{C}} \stackrel{(incl, -\mathrm{id})}{\longrightarrow} A_{\mathbb{C}})[-1]$$

(The notation of Burgos is $A^*(p)_{\mathrm{D}}$). Moreover, let $\mathrm{D}'^*(A, p)$ be the complex defined by

$$\mathrm{D}'^*(A, p) := \begin{cases} A^{n-1}_{\mathbb{R}}(p-1) \cap \oplus_{a+b=n-1, a,b<p} A^{a,b} & n < 2p \\ A^n_{\mathbb{R}}(p) \cap \oplus_{a+b=n, a,b \geq p} A^{a,b} & n \geq 2p \end{cases}$$

The differential $d_{\mathrm{D}'}(x)$ of some $x \in \mathrm{D}'^n(A, p)$ is defined as $-proj(dx)$ $(n < 2p-1)$, $-2\partial\overline{\partial}x$ $(n = 2p-1)$, and dx $(n \geq 2p)$. Here d is the standard exterior derivative, and $proj$ denotes the projection onto the space of forms of the appropriate bidegrees.

By [10, Theorem 2.6], there is a natural, concretely given homotopy equivalence between the complexes $\mathrm{D}^*(A, p)$ and $\mathrm{D}'^*(A, p)$. The complexes $\mathrm{D}^*(A, *)$ carry products $- \cup_\alpha -$ which depend on an auxiliary parameter $\alpha \in [0, 1]$. For any two α, α', the corresponding products on the complexes are in a natural way homotopy equivalent, $\cup_0$, $\cup_1$ are associative and $\cup_{1/2}$

is graded commutative. Therefore, the induced product

$$\mathrm{H}^n(\mathrm{D}^*(A,p)) \otimes \mathrm{H}^{n'}(\mathrm{D}^*(A,p')) \to \mathrm{H}^{n+n'}(\mathrm{D}^*(A,p+p'))$$

is independent of α and (graded) commutative and associative. The complexes $\mathrm{D}'^*(A,*)$ are slightly better behaved in that they inherit a product (via the afore-mentioned homotopy equivalence) which is independent of α (on the level of the complexes), is graded commutative, and is associative up to a homotopy that can be described concretely. The induced product on cohomology groups agrees with the previous product [10, Theorem 3.3].

Still following Burgos, we now construct a Dolbeault algebra which computes Deligne cohomology. As before, we let V/k be a smooth variety, where k is a subfield of $\mathbb{R}$.

Definition 2.3.3. [10, Def. 1.2] We consider a good compactification of V, i.e. an open immersion $V \hookrightarrow \overline{V}$ into a smooth proper variety, with complement D a normal crossings divisor. For any such compactification, we let $E^*_{\overline{V}(\mathbb{C})}(\log D(\mathbb{C}))$ be the complex of C^∞ differential forms that have at most logarithmic poles along the divisor D (see *loc. cit.* for details). Now we let $E^*(V(\mathbb{C}))$ be the following Dolbeault algebra:

$$E^*(V(\mathbb{C})) := \varinjlim E^*_{\overline{V}(\mathbb{C})}(\log D(\mathbb{C})),$$

where the colimit is taken over all compactifications of the above type.

The complex conjugation acts on the differential forms, forms fixed under that are referred to as real forms and denoted $E^*_{\mathbb{R}}(V)$. Finally, the complex is filtered by

$$F^p E^*(V(\mathbb{C})) := \oplus_{a \geq p, a+b=*} E^{a,b}(V(\mathbb{C})).$$

We write

$$E^*(V) := (E^*(V(\mathbb{C})))^{F_\infty}$$

for the subspace of elements fixed under the F_∞-action.

The above data gives a Dolbeault algebra $E(V)$. Applying the two constructions above to this Dolbeault algebra gives two complexes $\mathrm{D}^*(E(V),p)$

and $D'^*(E(V), p)$, and there are canonical isomorphisms

$$H^n_D(V, \mathbb{R}(p)) = H^n(D^*(E(V), p)) = H^n(D'^*(E(V), p)).$$

A key point here is that C^∞-forms are fine sheaves, hence acyclic. See [10, Prop. 1.3.] for more details.

For our purposes, the point of the above construction is the following theorem:

Theorem 2.3.4. *(Burgos) There exists presheaves of complexes E_p on the category of smooth varieties over k, satisfying the following properties:*

1. *For every smooth variety V, we have $H^n_D(V, \mathbb{R}(p)) = H^n(E_p(V))$ for all n and p.*

2. *The presheaves E_p are equipped with products $E_p \otimes E_{p'} \to E_{p+p'}$ which are (1) graded commutative on the level of complexes, and (2) associative up to homotopy. These products induce the usual product on the Deligne cohomology groups.*

2.4 Arithmetic Chow groups and related constructions

Before giving our construction of Arakelov motivic cohomology, we briefly review the properties of other related constructions in the literature, for comparison.

2.4.1 Arithmetic Chow groups

In this section we recall a few facts related to arithmetic Chow groups in the sense of Gillet and Soulé. For more background on these groups, we refer to [61] and the original paper [30]. We also mention a different definition of arithmetic Chow groups due to Burgos [10], which agrees with the Gillet-Soulé groups for schemes with proper generic fiber.

We shall restrict attention to schemes over the bases $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, although the definitions make sense over any base which is a so called arithmetic ring, see [30, Section 3.1]. An arithmetic variety over one of these bases is by definition a regular scheme which is flat and of finite type over the base in question.

Let X be an arithmetic variety over the base S. An arithmetic cycle on X is defined to be a pair (Z, g) where Z is an algebraic cycle on X and g is a so called Green current for Z. This means that the equation

$$dd^c g + \delta_Z = \omega$$

is satisfied for some smooth differential form ω. This is an equality of currents on the complex manifold associated to X, where δ_Z is the Dirac current associated with the cycle Z.

The arithmetic Chow group $\widehat{\mathrm{CH}}^p(X)$ is defined to be the group of arithmetic cycles (Z, g) with Z of codimension p, modulo a certain equivalence relation. The subgroup $\overline{\mathrm{CH}}^p(X)$ (sometimes denoted $\widehat{\mathrm{CH}}^p(X)_0$) is defined in the same way, but using only the arithmetic cycles such that

$$dd^c g + \delta_Z = 0$$

Although we will not use it, we mention that Gillet and Soulé also define another group $\mathrm{CH}^p(\overline{X})$ which depends on the choice of a metric on the complex manifold associated to X, and we always have inclusions

$$\overline{\mathrm{CH}}^p(X) \subset \mathrm{CH}^p(\overline{X}) \subset \widehat{\mathrm{CH}}^p(X).$$

We now list some of the key properties of arithmetic Chow groups.

- For any morphism $f : X \to Y$ between projective arithmetic varieties, there is a pullback

  $$f^* : \widehat{\mathrm{CH}}^p(Y) \to \widehat{\mathrm{CH}}^p(X)$$

  which is compatible with composition. If we in addition require that

X, Y are equidimensional, and f is proper and generically smooth, then there is also a push-forward

$$f_* : \widehat{\mathrm{CH}}^p(X) \to \widehat{\mathrm{CH}}^{p-d}(Y)$$

where $d = \dim X - \dim Y$.

- Writing $\widehat{\mathrm{CH}}^p(X)_{\mathbb{Q}} := \widehat{\mathrm{CH}}^p(X) \otimes \mathbb{Q}$, there is a product

$$\widehat{\mathrm{CH}}^p(X) \otimes \widehat{\mathrm{CH}}^q(X) \to \widehat{\mathrm{CH}}^{p+q}(X)_{\mathbb{Q}}$$

- The groups $\widehat{\mathrm{CH}}^p(X)$ are *not* homotopy invariant, but the subgroups $\overline{\mathrm{CH}}(X)$ are.

- The arithmetic Chow groups receive so called arithmetic characteristic classes of hermitian vector bundles on X. These induce an isomorphism

$$\widehat{\mathrm{K}}_0(X) \otimes \mathbb{Q} \to \oplus_p \widehat{\mathrm{CH}}^p(X)$$

for any arithmetic variety X, where $\widehat{\mathrm{K}}_0(X)$ is the arithmetic K_0-group defined in [31].

2.4.2 Regulator constructions and higher arithmetic Chow groups

As mentioned in the introduction, it is natural to ask if there is also an arithmetic version of Bloch's higher Chow groups. Based on insights which seem to go back to Beilinson, Deligne and Soulé, such groups are expected to sit in a long exact sequence

$$\cdots \widehat{\mathrm{CH}}^q(X, m) \to \mathrm{CH}^q(X, m) \overset{r_{Be}}{\to} \mathrm{H}_{\mathrm{D}}^{2q-m}(X, \mathbb{R}(q)) \to \widehat{\mathrm{CH}}^q(X, m-1) \cdots$$

involving the Beilinson regulator r_{Be}. Given the idea of such a sequence, a natural approach to defining higher arithmetic Chow groups is to take the cohomology of the homotopy fiber of the Beilinson regulator. However, for this to make sense one has to lift the regulator to some category in which the notion of homotopy fiber makes sense.

Classically, the Beilinson regulator is defined as a map from some version of motivic cohomology of a variety over $\mathbb{Q}$ to some version of Deligne cohomology or Hodge cohomology of the same variety. There are several constructions of the regulator in the literature, including Gillet's general Chern class formalism [54], Huber's realizations formalism [36], Bloch's higher cycle class map [6], and a similar cycle map construction by Scholl and Deninger [23]. However, none of these constructions give an obvious lift as required. However, in recent years several different lifts of the regulator have been constructed by various authors. Here we give a brief review of the constructions which are most relevant for us.

The construction of Goncharov. In [33], Goncharov constructs a rather explicit regulator map from a complex computing higher Chow groups to a complex computing Deligne cohomology, although he did not prove that the induced map on cohomology groups agrees with the Beilinson regulator. The construction is valid for smooth and projective varieties over $\mathbb{C}$ or $\mathbb{R}$. The resulting definition of $\widehat{\mathrm{CH}}^p(X, n)$ gives back the Gillet-Soulé groups when $n = 0$. However, it was not clear that these higher arithmetic Chow groups were contravariantly functorial, and it was also unclear whether they could be equipped with a product structure.

The construction of Burgos and Feliu. Based on work in Feliu's thesis [25], Burgos and Feliu [11] define a regulator map using a zigzag of explicit maps of complexes, starting with a complex computing higher Chow groups and ending with a complex computing Deligne cohomology. This leads to a definition of higher arithmetic Chow groups with the following properties. The construction is valid for any smooth quasi-projective variety over an arithmetic ring which is also a field (like $\mathbb{R}$, $\mathbb{C}$ or a number field). This removes the restriction to projective varieties in Goncharov's construction. The regulator agrees with the Beilinson regulator. The higher arithmetic Chow groups are contravariantly functorial, and carry a product structure.

For $n \geq 1$, the groups $\widehat{\mathrm{CH}}^p(X,n)$ are homotopy invariant. Depending on which Deligne complex one works with, the groups $\widehat{\mathrm{CH}}^p(X,n)$ for $n = 0$ can be made to agree either with $\widehat{\mathrm{CH}}^p(X)$ or with $\overline{\mathrm{CH}}^p(X)$.

In [12], Burgos, Feliu and Takeda compare the Burgos-Feliu construction with the one of Goncharov.

Other related constructions Explicit regulator maps on the level of complexes have also been studied by Kerr, Lewis and Müller-Stach in [39] and [38], as well as by Lima-Filho in [42], but none of these give any attention to the homotopy fiber.

Chapter 3

Arakelov motivic cohomology

Everything in this chapter is joint work with Jakob Scholbach.

3.1 Etale descent for Deligne cohomology

We will need to know that Deligne cohomology satisfies etale descent.

Definition 3.1.1. Consider a presheaf L on $\mathbf{Sm}/S$ with values in complexes of R-modules. We say that L satisfies etale descent if for any object X in $\mathbf{Sm}/S$, the natural map

$$H^n(L(X)) \to \mathbb{H}^n(X, L_{et})$$

is a quasi-isomorphism. Here L_{et} is the etale sheafification of L, and the map is the map induced by sheafification, from Hom in the derived category of presheaves to Hom in the derived category of sheaves.

Proposition 3.1.2. *Deligne cohomology satisfies etale descent (and hence also Nisnevich descent).*

Proof. The descent statement is stable under quasi-isomorphisms of complexes of presheaves and cones of maps of such complexes. Therefore it is sufficient to show descent for the three constituent parts of $\mathrm{D}^*(E(X), p)$, namely $X \mapsto E^*(X)(p)_{\mathbb{R}}$, $X \mapsto F^p E^*(X)$, $X \mapsto E^*(X)$. Let $j : X_{\mathbb{C}} \to \overline{X_{\mathbb{C}}}$

be an open immersion into a smooth compactification such that $D_{\mathbb{C}} := \overline{X_{\mathbb{C}}}$ is a divisor with normal crossings. The inclusion

$$\Omega^*_{\overline{X_{\mathbb{C}}}}(\log D_{\mathbb{C}}) \subset E^*_X(\log D)$$

of holomorphic forms into C^∞-forms (both with logarithmic poles) yields quasi-isomorphisms of complexes of vector spaces

$$\mathrm{R}\Gamma\mathrm{R}j_*\mathbb{C} \to \mathrm{R}\Gamma\mathrm{R}j_*\Omega^*_{X(\mathbb{C})} \leftarrow \mathrm{R}\Gamma\Omega^*_{\overline{X_{\mathbb{C}}}}(\log D_{\mathbb{C}}) \to \Gamma E^*_{X(\mathbb{C})}(\log D)$$

that is compatible with both the real structure and the Hodge filtration [9, Theorem 2.1.], [21, 3.1.7, 3.1.8]. Here (R)Γ denotes the (total derived functor of the) global section functor on $\overline{X_{\mathbb{C}}}(\mathbb{C})$, i.e., with respect to the analytic topology. Therefore the cohomology groups of $E^*(X)$ are just $\mathrm{H}^*(X(\mathbb{C}), \mathbb{C})$. Betti cohomology is known to satisfy etale descent [19, Cor 16.2.6]. This remains true if we replace $E^*(X)$ by $E^*_{\mathbb{R}}(X)(p)$, which computes the cohomology groups $\mathrm{H}^*(X(\mathbb{C}), \mathbb{R}(p))$. Moreover, taking fixed points or (-1)-eigenspace of complex conjugation (or infinite Frobenius) is an exact functor, so doing so preserves the etale descent property. We thus have shown the descent for the Betti part and the non-filtered de Rham part of the Deligne cohomology. See also [20, 3.1.3] for a proof of etale descent of the algebraic de Rham complex Ω^*_X.

It remains to show etale descent for $X \mapsto F^p E^*(X)$. We divide this into two steps: first the Nisnevich descent, then a descent under finite Galois covers. The first part requires to show that for a distinguished square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

(i.e., cartesian such that $Y' \to Y$ is an open immersion, X/Y is etale and induces an isomorphism $(X\backslash X')_{\mathrm{red}} \to (Y\backslash Y')_{\mathrm{red}}$) the following holds: the sequence

$$\mathrm{H}^n(F^pE(Y)) \to \mathrm{H}^n(F^pE(Y')) \oplus \mathrm{H}^n(F^pE(X)) \to \mathrm{H}^n(F^pE(X')) \\ \to \mathrm{H}^{n+1}(F^pE(Y))$$

is exact. Indeed, $\mathrm{H}^n(F^pE_X(\log D))$ maps injectively into $\mathrm{H}^n(\overline{X}, \Omega^*\overline{X}(\log D)$, and the image is precisely the p-th filtration step of the Hodge filtration on $\mathrm{H}^n(\overline{X}, \Omega^*_{\overline{X}}(\log D)) = \mathrm{H}^n(X, \mathbb{C})$. Similarly for X' etc., so that the exactness of the above sequence follows from the one featuring the Betti cohomology groups $\mathrm{H}^*(-(\mathbb{C}), \mathbb{C})$ of Y, $Y' \sqcup X$ and X', respectively, together with the strictness of the Hodge filtration [21, Th. 1.2.10]. This shows Nisnevich descent for the Hodge filtration [19, Theorem 3.3.2]. Secondly, for any scheme X and a Galois cover $Y \to X$ with group G, the pullback map into the G-invariant subspace

$$\mathrm{H}^n(F^pE(X)) \to \mathrm{H}^n(F^pE(Y)^G)$$

is an isomorphism. To see this we first use the etale descent of de Rham cohomology: the similar statement holds for $E(-)$ instead of $F^pE(-)$. We work with $\mathbb{Q}$-coefficients, so taking G-invariants is an exact functor, hence $\mathrm{H}^n(F^pE(Y)^G) = (\mathrm{H}^n(F^pE(Y)))^G = (F^p\mathrm{H}^n_{\mathrm{dR}}(Y))^G = F^p \cap \mathrm{H}^n_{\mathrm{dR}}(Y)^G$, the last equality by functoriality of the Hodge filtration. Then, again using the strictness of the Hodge filtration, the claim follows. Hence (this uses $\mathbb{Q}$-coefficients) by [19, Theorem 3.3.22], the presheaf $X \mapsto F^pE(X)$ has etale descent. □

3.2 Constructing a Deligne spectrum

Again, let k be a subfield of $\mathbb{R}$. In this section, we construct a spectrum over $\operatorname{Spec} k$, i.e. an object in the motivic stable homotopy category over $\operatorname{Spec} k$, which represents Deligne cohomology. The method is a slight variation of the method of Cisinski and Déglise used in [20] to construct a spectrum for any mixed Weil cohomology (or more generally what they call a stable cohomology theory), such as algebraic or analytic de Rham cohomology, Betti cohomology, and (geometric) etale cohomology. The main differences compared to their setting is that the Tate twist on Deligne cohomology groups is not an isomorphism of vector spaces.

Write $S = \operatorname{Spec} k$. In this section we assume that we are given complexes E_p, functorial on the category of smooth k-varieties, such that the following holds: There are products $\mu = \mu_{p,p'} : E_p \otimes E_{p'} \to E_{p+p'}$, and the complexes E_p compute Deligne cohomology, in the sense that $H^n_D(X, p) = H^n(E_p(X))$ for all X, n and p, with the products in cohomology induced by μ. We also assume μ to be strictly commutative, and associative at least up to homotopy.

Remark 3.2.1. In the present article, the complex E_p will be the Burgos complex $\mathrm{D}'^*(X, p)$ introduced earlier, but one could also consider other complexes computing (possibly other versions of) Deligne cohomology. Because the complexes $\mathrm{D}'^*(X, p)$ have a product which is associative only up to homotopy, we can only construct a weak ring spectrum, i.e. a monoid in the stable homotopy category rather than in the underlying category of spectra. If one could find Deligne complexes with strictly associative and commutative products, the construction presented here would give a strict ring spectrum. We recently learnt from Burgos that such complexes can actually be constructed which compute weak Hodge cohomology, using a modification of [15, 6.2]. Weak Hodge cohomology (see [4, 3.13], [48, 7.1]) agrees with Deligne cohomology at least for smooth projective varietes, and if there are varieties for which they differ, weak Hodge cohomology would be the better one from the perspective of Scholbach's conjecture. A proper treatment of the weak Hodge cohomology viewpoint will be included in [34] - in this thesis we just work with the Deligne complexes given in the previous chapter.

First we note that the complexes E_p represent Deligne cohomology in the effective category:

Lemma 3.2.2. *In the model category underlying $D^{\mathrm{eff}}_{\mathbb{A}^1,\mathbb{Q}}(S)$, the presheaf* *complexes E_p are fibrant and in particular $\mathbb{A}^1$-local. Moreover,*

$$\mathrm{Hom}_{D^{\mathrm{eff}}_{\mathbb{A}^1,\mathbb{Q}}}(\mathbb{Q}(X), E_p[n]) = \mathrm{H}^n_{\mathrm{D}}(X, p)$$

for any smooth k-variety X.

Proof. By Prop 2.2.4, the fibrant objects are the ones which are $\mathbb{A}^1$-invariant and Nisnevich local, i.e. any such object satisfies Nisnevich descent. The complexes E_p are fibrant since Deligne cohomology is $\mathbb{A}^1$-invariant and satisfies Nisnevich descent. Recall that a complex K of Nisnevich sheaves is $\mathbb{A}^1$-local if

$$\mathrm{H}^*_{\mathrm{Nis}}(X, K) = \mathrm{H}^*_{\mathrm{Nis}}(X \times \mathbb{A}^1, K).$$

Clearly any fibrant complex of Nisnevich sheaves is $\mathbb{A}^1$-local.

According to Prop 2.2.5, the localization functor

$$\phi : \mathbf{D}(\mathbf{Shv}_{\mathrm{Nis}}(\mathbf{Sm}/S, \mathbb{Q})) \to \mathbf{D}^{\mathrm{eff}}_{\mathbb{A}^1,\mathbb{Q}}(S)$$

has a fully faithful right adjoint ψ, whose essential image is given by the $\mathbb{A}^1$-local complexes. In particular, the complexes E_p are such that $\psi\phi(E_p) \cong E_p$ (isomorphism in the derived category of sheaves). Thus there is a chain of canonical isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}^{\mathrm{eff}}_{\mathbb{A}^1,\mathbb{Q}}}(\phi\mathbb{Q}(X), \phi(E_p)[n]) &= \mathrm{Hom}_{\mathbf{D}(\mathbf{Shv})}(\mathbb{Q}(X), \psi\phi E_p[n]) \\ &= \mathrm{Hom}_{\mathbf{D}(\mathbf{Shv})}(\mathbb{Q}(X), E_p[n]) \\ &= \mathrm{H}^n_{\mathrm{Nis}}(X, (E_p)_{\mathrm{Nis}}) \\ &= \mathrm{H}^n(E_p(X)) \\ &= \mathrm{H}^n_{\mathrm{D}}(X, p) \end{aligned}$$

□

Definition 3.2.3. Let $\mathcal{E}$ be the following $\mathbb{Q}(1)$-spectrum: the p-th component of the spectrum is simply the complex E_p, with trivial action of the symmetric group Σ_p. The bonding maps $\sigma_p : \mathbb{Q}(1) \otimes E_p \to E_{p+1}$ are constructed as follows. Let $K \in \mathrm{Pic}(\mathbb{P}^1)$ be the generator $\mathcal{O}(-1)$. Let

$c_1 : \mathrm{Pic}(\mathbb{P}^1) \to H^2_{\mathrm{D}}(\mathbb{P}^1, 1)$ be the first Chern class map. By the vanishing of $\mathrm{Hom}_{\mathbf{D}^{\mathrm{eff}}_{\mathbb{A}^1,\mathbb{Q}}}(\mathbb{Q}, E_1[2]) = \mathrm{H}^2_{\mathrm{D}}(\mathbb{Q}, 1)$ we have

$$\begin{aligned}
\mathrm{H}^2_{\mathrm{D}}(\mathbb{P}^1, 1) &= \mathrm{Hom}_{\mathbf{D}^{\mathrm{eff}}_{\mathbb{A}^1,\mathbb{Q}}}(\mathrm{M}(\mathbb{P}^1), E_1[2]) \\
&= \mathrm{Hom}_{\mathbf{D}^{\mathrm{eff}}_{\mathbb{A}^1,\mathbb{Q}}}(\mathbb{Q}(1)[2], E_1[2]) \\
&= \mathrm{Hom}_{\mathbf{D}^{\mathrm{eff}}_{\mathbb{A}^1,\mathbb{Q}}}(\mathbb{Q}(1), E_1).
\end{aligned}$$

Let β be the image of $c_1(K)$ under this chain of identifications. Because $\mathbb{Q}(1)$ is cofibrant [20, 1.3.1 & 1.1.15] and E_1 is fibrant, in the model category underlying $\mathbf{D}^{\mathrm{eff}}_{\mathbb{A}^1,\mathbb{Q}}$, the map β in can be lifted to an actual map in the model category, which we call β'. Using this map, we construct bonding maps σ_p as the following composition:

$$\mathbb{Q}(1) \otimes E_p \overset{\beta' \otimes id}{\to} E_1 \otimes E_p \overset{\mu_{1,p}}{\to} E_{p+1}$$

This gives a spectrum $\mathcal{E}$. As an object of $\mathbf{D}_{\mathbb{A}^1,\mathbb{Q}}(\mathbb{Q})$, this is independent of the choice of the lift β'.

We equip $\mathcal{E}$ with the structure of a monoid as follows: the product $\mu : \mathcal{E} \wedge \mathcal{E} \to \mathcal{E}$ is induced by the products $\mu_{p,p'} : E_p \otimes E_{p'} \to E_{p+p'}$. The unit map $\Sigma^\infty \mathbb{Q} \to \mathcal{E}$ is defined by a sequence of maps $\eta_p : \mathbb{Q}(p) \to E_p$ constructed as follows. On the 0-th level, we define $\eta_0 : \mathbb{Q}(0) \to E_0$ to be the unit of E_0 viewed as a differential graded algebra. The other components of the unit map are then given recursively as $\eta_p := \sigma_{p-1} \circ \eta_{p-1}(1) : \mathbb{Q}(p) \to E_p$.

If the products μ are both commutative and associative on the nose on the level of complexes, this construction gives a strict commutative ring spectrum, i.e. a commutative monoid object in the category of symmetric $\mathbb{Q}(1)$-spectra. (For the precise diagrams which have to be checked, see [58]; since all our symmetric group actions are trivial these diagrams boil down to commutativity and associativity of the products μ). If like in the case of the Burgos complexes, the products are commutative on the nose but associative only up to homotopy the construction still gives a spectrum, but with the structure of a commutative monoid object only in the homotopy category of spectra, i.e. what we call a weak ring spectrum (or just a ring spectrum in the terminology of [19]). (In this case all the same diagrams are commutative on the model category level except for the associativity diagram, which becomes commutative in **SH**).

Theorem 3.2.4. *Recall that* $S = \mathrm{Spec}\,(k)$ *where* k *is a subfield of* $\mathbb{R}$*. Let* $\mathcal{E}_{\mathrm{D}}$ *be the homotopy ring spectrum constructed from the Burgos complexes* $\mathrm{D}'^{*}(X,p)$ *according to the above recipe. Then* $\mathcal{E}_{\mathrm{D}}$ *represents Deligne cohomology in* $\mathbf{D}_{\mathbb{A}^1,\mathbb{Q}}(S)$*, i.e. for any smooth variety* X *over* k*, we have*

$$\mathrm{Hom}_{\mathbf{D}_{\mathbb{A}^1,\mathbb{Q}}}(\mathbb{Q}(X), \mathcal{E}_{\mathrm{D}}(p)[n]) = \mathrm{H}^n_{\mathrm{D}}(X, \mathbb{R}(p)).$$

Proof. Lemma 3.2.2 and Prop 2.2.11 together imply that it is enough to prove that $\mathcal{E}_{\mathrm{D}}$ is a weak Ω^∞-spectrum, i.e. that

$$E_{p-1} \cong \mathbf{RHom}(\mathbb{Q}(1), E_p)$$

for all $p \geq 1$ (isomorphism in D^{eff}). For this, it suffices to show that for any smooth X and any n, we have isomorphisms

$$\mathrm{Hom}(\mathbb{Q}(X), \mathbf{RHom}(\mathbb{Q}(1), E_p)[n]) \cong \mathrm{Hom}(\mathbb{Q}(X), E_{p-1}[n]) \qquad (3.1)$$

(all Hom groups taken in the effective category). On one hand, the weak Künneth formula gives:

$$H^{n+2}(X \times \mathbb{P}^1, p) = H^{n+2}(X,p) \oplus H^n(X, p-1).$$

On the other hand, Lemma 3.2.2 implies

$$\begin{aligned} H^{n+2}(X \times \mathbb{P}^1, p) &= \mathrm{Hom}(\mathbb{Q}(X) \otimes (\mathbb{Q} \oplus \mathbb{Q}(1)[2]), E_p[n+2]) \\ &= \mathrm{Hom}(\mathbb{Q}(X), E_p[n+2]) \oplus \mathrm{Hom}(\mathbb{Q}(X)(1)[2], E_p[n+2]) \\ &= \mathrm{H}^{n+2}(X,p) \oplus \mathrm{Hom}(\mathbb{Q}(X)(1)[2], E_p[n+2]) \end{aligned}$$

and using that the weak Künneth decomposition is induced by cup product, we can conclude that

$$\mathrm{Hom}(\mathbb{Q}(X)(1)[2], E_p[n+2]) \cong H^n(X, p-1)$$

(functorially in X). Now using this equality, we compute:

$$\begin{aligned}
\mathrm{Hom}(\mathbb{Q}(X), \mathbf{RHom}(\mathbb{Q}(1), E_p)[n]) &= \mathrm{Hom}(\mathbb{Q}(X) \otimes \mathbb{Q}(1), E_p[n]) \\
&= \mathrm{Hom}(\mathbb{Q}(X) \otimes \mathbb{Q}(1)[2], E_p[n+2]) \\
&= H^n(X, p-1) \\
&= \mathrm{Hom}(\mathbb{Q}(X), E_{p-1}[n])
\end{aligned}$$

which proves equation (3.1), so $\mathcal{E}_{\mathrm{D}}$ is indeed a weak Ω^∞-spectrum. □

3.3 First definition of Arakelov motivic cohomology

Using a deep theorem of Cisinski and Déglise, we can define a regulator map r from the Beilinson spectrum H_{B} to the Deligne spectrum.

Theorem 3.3.1. *[19, Corollary 13.2.15 and Corollary to Thm 13, p. 7] Let S be a noetherian scheme of finite Krull dimension, and let $\mathcal{E}$ be weak ring spectrum with rational coefficients over S, i.e. a commutative monoid object in $\mathbf{D}_{\mathbb{A}^1,\mathbb{Q}}(S)$. Then the following are equivalent:*

- *$\mathcal{E}$ is an H_{B}-module in $\mathbf{D}_{\mathbb{A}^1,\mathbb{Q}}$, i.e. it lies in $\mathbf{DM}_{\mathrm{B}}(S)$.*
- *$\mathcal{E}$ satisfies etale descent.*
- *$\mathcal{E}$ admits a unique structure of H_{B}-algebra.*

Corollary 3.3.2. *The Deligne cohomology spectrum $\mathcal{E}_{\mathrm{D}} \in \mathbf{D}_{\mathbb{A}^1,\mathbb{Q}}(S)$ has a unique structure of H_{B}-algebra. This gives a map $\mathrm{H}_{\mathrm{B},\mathrm{Spec}\,\mathbb{Q}} \to \mathcal{E}_{\mathrm{D}}$ in $\mathbf{D}_{\mathbb{A}^1,\mathbb{Q}}(S)$, which we denote by r.*

Proof. By construction, $\mathcal{E}_{\mathrm{D}}$ is a commutative ring spectrum. By Prop 3.1.2, it satisfies etale descent, so Theorem 3.3.1 applies. □

In [34] we will prove that the induced map on cohomology groups agrees with the classical Beilinson regulator.

Using this regulator map, we can now define the Arakelov motivic cohomology spectrum. For applications to the study of special values, we are primarily interested in Arakelov motivic cohomology for arithmetic schemes and motives over $\operatorname{Spec}\mathbb{Z}$. However, we will also consider a version for varieties and motives over $\mathbb{Q}$, following the example of Gillet-Soulé arithmetic Chow groups, which are defined for both kinds of base schemes, and with a view to comparison theorems with the higher arithmetic Chow groups of Burgos and Feliu.

Recall that $\mathrm{H}\mathbb{Z}_S$ and $\mathrm{H}\mathbb{Q}_S$ denote the spectra which represent motivic cohomology with $\mathbb{Z}$ and $\mathbb{Q}$-coefficients over the base scheme S. Recall also that H_{B} is the Beilinson spectrum from Section 2.2.1, where if necessary we also indicate the base scheme with a subscript ($\mathrm{H}_{\mathrm{B},\operatorname{Spec}\mathbb{Q}}$, $\mathrm{H}_{\mathrm{B},\operatorname{Spec}\mathbb{Z}}$ etc). Finally, we write $\eta : \operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}$ for the generic point of $\operatorname{Spec}\mathbb{Z}$; by Theorem 2.2.16 it induces in particular an adjoint pair of functors

$$\eta^* : \mathbf{SH}(\operatorname{Spec}\mathbb{Z}) \rightleftarrows \mathbf{SH}(\operatorname{Spec}\mathbb{Q}) : \eta_*.$$

Definition 3.3.3. We define the Arakelov motivic cohomology spectrum $\widehat{\mathrm{H}\mathbb{Z}}$ (or $\widehat{\mathrm{H}\mathbb{Z}}_{\operatorname{Spec}\mathbb{Z}}$) in $\mathbf{SH}(\operatorname{Spec}\mathbb{Z})$ as the homotopy fiber of the following zigzag of morphisms:

$$\widehat{\mathrm{H}\mathbb{Z}} := \operatorname{hofib}\big(\mathrm{H}\mathbb{Z}_{\operatorname{Spec}\mathbb{Z}} \to \mathrm{H}\mathbb{Q}_{\operatorname{Spec}\mathbb{Z}} \xleftarrow{\sim} \mathrm{H}_{\mathrm{B},\operatorname{Spec}\mathbb{Z}} \to \eta_*\mathrm{H}_{\mathrm{B},\operatorname{Spec}\mathbb{Q}} \to \eta_*\mathcal{E}_{\mathrm{D}}\big).$$

Here the second map is a weak equivalence by [19, Corollary 15.1.6], the third is the adjoint of the isomorphism $\eta^*\mathrm{H}_{\mathrm{B},\operatorname{Spec}\mathbb{Z}} \to \mathrm{H}_{\mathrm{B},\operatorname{Spec}\mathbb{Q}}$, and the fourth is the regulator map defined above.

Having defined this spectrum, we can apply the recipe of Section 2.2.3 to define its associated theories of cohomology, cohomology with compact support, homology, and Borel-Moore homology. However, because we are over an arithmetic base scheme, something unexpected happens at least in the case of cohomology, in that there is also another distinct candidate for a cohomology theory associated to a spectrum. In this chapter we give the usual definition of cohomology, and in the next chapter we discuss the other version, which is better for some purposes.

Definition 3.3.4. Let Λ be a ring such that $\mathbb{Z} \subset \Lambda \subset \mathbb{R}$. For any (separated) scheme $f : X \to \operatorname{Spec} \mathbb{Z}$ of finite type, we can define *Arakelov motivic cohomology* of X with coefficients in Λ by the following formula:

$$\widehat{\mathrm{H}}^n(X, \Lambda(p)) := \mathrm{Hom}_{\mathbf{SH}(\operatorname{Spec} \mathbb{Z})}(\mathbf{1}, f_* f^* \widehat{\mathrm{H}\mathbb{Z}}(p)[n]) \otimes \Lambda.$$

We remark that in the case where f is smooth, this definition agrees with the alternative one given in chapter 4.

Definition 3.3.5. For a variety V over $\mathbb{Q}$, we define Arakelov motivic cohomology by

$$\widehat{\mathrm{H}}^n(V, \Lambda(p)) := \mathrm{Hom}_{\mathbf{SH}(\operatorname{Spec} \mathbb{Q})}(\mathbf{1}, f_* f^* \widehat{\mathrm{H}\mathbb{Z}}_{\operatorname{Spec} \mathbb{Q}}(p)[n]).$$

where

$$\widehat{\mathrm{H}\mathbb{Z}}_{\operatorname{Spec}} \mathbb{Q} := \operatorname{hofib}\left(\mathrm{H}\mathbb{Z}_{\operatorname{Spec} \mathbb{Q}} \to \mathrm{H}\mathbb{Q}_{\operatorname{Spec} \mathbb{Q}} \xleftarrow{\sim} \mathrm{H}_{\mathrm{B}, \operatorname{Spec} \mathbb{Q}} \to \mathcal{E}_{\mathrm{D}}\right).$$

Note that for the purposes of understanding the Beilinson conjectures, we are interested in the version over $\operatorname{Spec} \mathbb{Z}$ even though we look at special values of L-functions of varieties over $\mathbb{Q}$. One reason is that after tensoring with $\mathbb{Q}$, the term $\mathrm{H}_{\mathrm{B}, \operatorname{Spec} \mathbb{Z}}$ which occurs only in the definition over $\operatorname{Spec} \mathbb{Z}$, captures what is classically referred to as the "K-theory of an integral model" (or rather a suitable Adams-graded piece of the K-theory). Another related reason is seen in the reformulation of the Beilinson conjectures given by Scholbach, which we discuss in Chapter 5. This conjecture gives some evidence that the natural setting for the Beilinson conjectures is that of motives over $\operatorname{Spec} \mathbb{Z}$ rather than over $\operatorname{Spec} \mathbb{Q}$.

3.4 Basic properties

As a first consequence of the definition we get a version of the Beilinson localization sequence described in the introduction. The severe restriction to smooth schemes disappears when we consider the alternative definition in Chapter 4.

Proposition 3.4.1. *Let* $f : X \to \operatorname{Spec}\mathbb{Z}$ *be a smooth arithmetic scheme ("arithmetic" meaning flat and of finite type over* $\operatorname{Spec}\mathbb{Z}$*). Then we have the following long exact sequence:*

$$\cdots \widehat{\mathrm{H}}^n(X, \mathbb{Z}(p)) \to \mathrm{H}^n_{\mathrm{M}}(X, \mathbb{Z}(p)) \to \mathrm{H}^n_{\mathrm{D}}(X, \mathbb{R}(p)) \to \widehat{\mathrm{H}}^{n+1}(X, \mathbb{Z}(p)) \cdots$$

involving Arakelov motivic cohomology of X *together with the regulator map from naive motivic cohomology of* X *to Deligne cohomology of the generic fiber of* X*.*

Proof. Recall that the distinguished triangles in **SH** are by definition those coming from fiber sequences. Hence by definition of $\widehat{\mathrm{H}\mathbb{Z}}$, there is a distinguished triangle $\widehat{\mathrm{H}\mathbb{Z}} \to \mathrm{H}\mathbb{Z} \to \eta_* \mathcal{E}_{\mathrm{D}}$. Applying $\mathrm{Hom}_{\mathbf{SH}(\operatorname{Spec}\mathbb{Z})}(\mathbf{1}, f_* f^*(-))$ gives a long exact sequence. To conclude, we have to identify these Hom groups with the above cohomology groups. For the first two groups, this is just the definitions. For the Deligne cohomology groups, we use the notation given by the diagram of generic fibers

$$\begin{array}{ccc} X_\eta & \xrightarrow{\eta'} & X \\ {\scriptstyle f_\eta}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}\mathbb{Q} & \xrightarrow[\eta]{} & \operatorname{Spec}\mathbb{Z} \end{array}$$

and the following chain of isomorphisms, in which we use adjointness properties of the functors involved, relative purity relating f^* and $f^!$, and base change (all part of the six functors formalism).

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Spec}\,\mathbb{Z}}(\mathbf{1}, f_*f^*\eta_*\mathcal{E}_{\mathrm{D}}) &\cong \mathrm{Hom}_X(f^*\mathbf{1}, f^*\eta_*\mathcal{E}_{\mathrm{D}}) \\
&\cong \mathrm{Hom}_X(\mathbf{1}, f^!\eta_*\mathcal{E}_{\mathrm{D}}(-d)[-2d]) \\
&\cong \mathrm{Hom}_X(\mathbf{1}, \eta'_*f_\eta^!\mathcal{E}_{\mathrm{D}}(-d)[-2d]) \\
&\cong \mathrm{Hom}_{X_\eta}(\eta'^*\mathbf{1}, f_\eta^!\mathcal{E}_{\mathrm{D}}(-d)[-2d]) \\
&\cong \mathrm{Hom}_{X_\eta}(\mathbf{1}, f_\eta^*\mathcal{E}_{\mathrm{D}}) \\
&\cong \mathrm{Hom}_{\mathrm{Spec}\,\mathbb{Q}}(\mathbf{1}, f_{\eta*}f_\eta^*\mathcal{E}_{\mathrm{D}}) \\
&\cong \mathrm{H}^n_{\mathrm{D}}(X, p)
\end{aligned}$$

Here d is the relative dimension of f, the subscripts (say Y) on a Hom group indicates that we take Hom in $\mathbf{SH}(Y)$, or equivalently (by Theorem 2.2.21) in $\mathbf{DM}_{\mathrm{B}}(Y)$. Note that we used that f is smooth in order to apply relative purity. As far as I can see, there is no reason to think that the same identification would be true at all in case f is not smooth. □

For the conjecture of Scholbach, we are interested in the above sequence tensored with $\mathbb{R}$. In this case the naive motivic cohomology does agree[1] with usual motivic cohomology unconditionally, which if X is regular can also be expressed as a suitable Adams-graded piece of the algebraic K-theory of X.

The Arakelov motivic cohomology groups satisfy a number of formal properties which are true simply because they hold for any cohomology theory representable by a spectrum. Many of these properties would have been difficult to establish had we used a different approach to the construction. The list of such properties include:

- $\mathbb{A}^1$-homotopy invariance.
- Contravariant functoriality for arbitrary morphisms.
- Mayer-Vietoris long exact sequence (as a special case of Nisnevich descent).

[1] When tensoring Voevodsky's motivic cohomology groups $\mathrm{H}^n_{\mathrm{M}}(X, \mathbb{Z}(p))$ with $\mathbb{Q}$ or $\mathbb{R}$, we recover Voevodsky's motivic cohomology with $\mathbb{Q}$ or $\mathbb{R}$-coefficients, *provided we work over a regular base or a base of characteristic zero.*

- Localization long exact sequences associated to the two localization triangles of Theorem 2.2.17

If we work with rational or real coefficients, the groups also satisfy h-descent (and hence also etale, cdh, qfh and proper descent). This follows from the fact that for $\Lambda = \mathbb{Q}$ or $\Lambda = \mathbb{R}$ the spectra $\widehat{\mathrm{H}\mathbb{Z}} \otimes \Lambda$ are H_{B}-modules (and MGL-modules), so we can apply theorem [19, Thm 15.1.3] of Cisinski and Déglise. This module structure is in turn a consequence of the the following simple observation.

Lemma 3.4.2. *If* $\mathbf{R}$ *is a strict ring spectrum in* $\mathbf{SH}$*, and we take a homotopy fiber of maps which are* $\mathbf{R}$*-module homomorphisms, the result is independent of whether we take the homotopy fiber inside* $\mathbf{SH}$ *or inside the homotopy category of* $\mathbf{R}$*-modules.*

Proof. The forgetful functor from $\mathbf{R}$-modules to $\mathbf{SH}$ is a right Quillen functor, (the left Quillen functor being $E \mapsto \mathbf{R} \otimes E$), so it preserves homotopy limits and in particular homotopy fibers. □

There are other desirable properties which are not automatic consequences of general facts in motivic stable homotopy theory. One is the question of product structure, another is the question of push-forward functoriality for proper morphisms, and a third is the question of whether there is a theory of Chern classes. Push-forwards are discussed in Section 3.5 and Chapter 4. In Chapter 6, we construct a product structure. We expect that there should also be a theory of Chern classes, and will come back to this in a future paper.

3.5 Pushforward functoriality

As discussed in the introduction, one could hope that the Arakelov motivic cohomology and arithmetic K-theory satisfy some form of higher arithmetic Riemann-Roch square. However, to write down such a square one must have push-forward functoriality for proper morphisms. We have the following result.

Proposition 3.5.1. *Let Λ be either $\mathbb{Q}$ or $\mathbb{R}$. Let $f : X \to Y$ be a smooth projective morphism of relative dimension d between arithmetic schemes. Then there are functorial pushforward maps*

$$f_* : \widehat{\mathrm{H}}^n(X, \Lambda(p)) \to \widehat{\mathrm{H}}^{n-2d}(Y, \Lambda(p-d)).$$

Proof. This proof relies only on the fact that $\widehat{\mathrm{H}\Lambda}$ is a H_{B}-module for $\Lambda = \mathbb{Q}$ or $\mathbb{R}$. Let p_X and p_Y be the structural morphisms of X and Y respectively. It is enough to construct a map $p_{X*}p_X^* = p_{Y*}f_*f^*p_Y^* \to p_{Y*}p_Y^*(d)[2d]$. But $f_* = f_!$ since f is proper, and by Theorem 2.2.18, we have $f^* = f^!(-d)[-2d]$ since f is smooth and quasi-projective, so the counit $f_!f^! \to id$ gives the desired map. □

In the next chapter, we show that we get better pushforward functoriality with the second version of Arakelov motivic cohomology.

Chapter 4

An alternative definition of Arakelov motivic cohomology

The content of this chapter is joint work in progress with Jakob Scholbach.

We recently realized that it is possible to give a slightly different definition of Arakelov motivic cohomology which at least for some purposes is much better. In this chapter, we give this definition and discuss its advantages, with only partial proofs.

4.1 A surprise over arithmetic base schemes

When working with motivic stable homotopy theory over a field k, every spectrum $E \in \mathbf{SH}(\mathrm{Spec}\ k)$ gives rise to a (bigraded) cohomology theory defined on all schemes of finite type over k. For such a scheme $f : V \to \mathrm{Spec}\, k$, the definition of these cohomology groups is

$$E^n(V, i) = \mathrm{Hom}_{\mathbf{SH}(\mathrm{Spec}\, k)}(\mathbf{1}, f_* f^* E(i)[n])$$

where $\mathbf{1}$ is the unit object in $\mathbf{SH}(\mathrm{Spec}\, k)$. If k is replaced by an arithmetic base scheme S (like $\mathrm{Spec}\, \mathbb{Z}$), the same definition makes sense and defines a cohomology theory for all schemes of finite type over S. However, a new very interesting phenomenon appears: There is also another sensible way to associate a cohomology theory to a spectrum. This alternative definition is given by the following formula, where $f : X \to S$ is a separated morphism of finite type:

$$E^n(V, i) = \mathrm{Hom}_{\mathbf{SH}(S)}(f_! f^! \mathbf{1}, E(i)[n])$$

According to [18], the following is true: over a field, both definitions give the same cohomology theory, at least if the field admits resolution of singularities, or in general if the theory is orientable with rational coefficients. Presumably this is the reason that the second definition is not discussed in the literature. Under very restrictive hypotheses, the two definitions will agree also over an arithmetic base, for example if we restrict to smooth f, or if E is very nice (for example, it could be that they always agree if E is an orientable theory with rational coefficents, which satisfies absolute purity). However, if we look for example at the class of all regular arithmetic schemes and a general E, there is no reason to think that the definitions agree. In particular, there is no reason to expect the two definitions to agree for $\eta_* \mathcal{E}_{\mathrm{D}}$ or for $\widehat{\mathrm{H}\mathbb{Z}}$. As far as I understand the two definitions agree for the Beilinson spectrum, by work of Cisinski and Déglise, but I am not sure what to expect for example in the case of algebraic K-theory or motivic cohomology with integral coefficients.

Anyway, for us the moral of all this is that because we work over an arithmetic base, we have two distinct reasonable candidates for Arakelov motivic cohomology, which will agree for smooth schemes but (most likely) not in general. In this chapter we discuss properties of the second definition. Recall the spectrum $\widehat{\mathrm{H}\mathbb{Z}}$ defined above.

Definition 4.1.1 (New definition of Arakelov motivic cohomology)**.** Let Λ be a ring such that $\mathbb{Z} \subset \Lambda \subset \mathbb{R}$. For any separated scheme $f : X \to \mathrm{Spec}\,\mathbb{Z}$ of finite type, we define *Arakelov motivic cohomology* of X with coefficients in Λ by the following formula:

$$\widehat{\mathrm{H}}^n(X, \Lambda(p)) := \mathrm{Hom}_{\mathbf{SH}(\mathrm{Spec}\,\mathbb{Z})}(f_! f^! \mathbf{1}, \widehat{\mathrm{H}\mathbb{Z}}(p)[n]) \otimes \Lambda.$$

4.2 Advantages of the new definition

Like the first definition, this cohomology definition is also automatically contravariantly functorial for all morphisms and $\mathbb{A}^1$-invariant. However, there are also several significant advantages, which we now list.

The Beilinson localization sequence for all regular schemes. Recall that with the first definition we obtained the long exact sequence

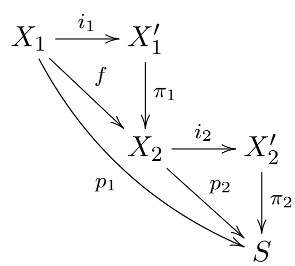
$$\cdots \widehat{\mathrm{H}}^n(X, \mathbb{Z}(p)) \to \mathrm{H}^n_{\mathrm{M}}(X, \mathbb{Z}(p)) \to \mathrm{H}^n_{\mathrm{D}}(X, \mathbb{R}(p)) \to \widehat{\mathrm{H}}^{n+1}(X, \mathbb{Z}(p)) \cdots$$

only in the case where X is smooth over $\operatorname{Spec}\mathbb{Z}$. With the new definition, we do get a version of this sequence for all X with smooth generic fiber, in particular for all regular X. The reason is that with the new definition, the spectrum $\eta_*\mathcal{E}_{\mathrm{D}}$ actually represents Deligne cohomology whenever the generic fiber is smooth. The version of motivic cohomology in our new exact sequence will be $\mathrm{Hom}_{\mathbf{SH}(\operatorname{Spec}\mathbb{Z})}(f_!f^!\mathbf{1}, \mathrm{H}\mathbb{Z}(p)[n])$, and we are not sure if this can identified with usual motivic cohomology, but as soon as we tensor with $\mathbb{Q}$ or $\mathbb{R}$, this identification should be true.

Compatibility with the definition for motives. In chapter 5, we will define Arakelov motivic cohomology for motives over $\operatorname{Spec}\mathbb{Z}$, which is crucial for making the connection between Arakelov motivic cohomology and Scholbach's reformulation of the Beilinson conjectures. It seems natural to hope for a functor from finite type schemes to motives, such that the definition of Arakelov motivic cohomology for schemes becomes just a special case of the definition for motives. The problem is that the first definition of Arakelov motivic cohomology for schemes does not seem to factor through the category of motives (unless we restrict to smooth schemes over $\operatorname{Spec}\mathbb{Z}$). With the new definition however, we do get such a factorization, if we define the motive of $f : X \to \operatorname{Spec}\mathbb{Z}$ to be simply $f_!f^!\mathbf{1}$ (this formula defines a functor from finite type schemes over $\operatorname{Spec}\mathbb{Z}$ into $\mathbf{DM}_{\text{B}}(\operatorname{Spec}\mathbb{Z})$, and also into $\mathbf{SH}(\operatorname{Spec}\mathbb{Z})$).

Better pushforwards. With the first definition, we could obtain pushforward functoriality only for smooth projective morphisms, and the smoothness here is a very restrictive condition. With the new definition, we still have pushforwards for smooth projective morphisms between arbitrary arithmetic schemes (for the same reasons as before), but the following proposition shows that with the new definition, we get much more, in particular we get pushforwards for any projective morphism between regular projective arithmetic schemes. If this type of functoriality can be treated locally, it should be possible to extend the argument to any proper map between regular arithmetic schemes, using that any map between regular arithmetic schemes is l.c.i.

Proposition 4.2.1. *Let $f : X_1 \to X_2$ be a proper map between two regular schemes. We assume that the situation is as follows*



Here i_1 and i_2 are closed immersions, π_1 and π_2 are smooth quasi-projective maps and X_1' and X_2' are regular schemes. For simplicity of notation we also assume all schemes connected. Then there is a functorial pushforward map

$$f_* : \widehat{\mathrm{H}}^n(X_1, \Lambda(p)) \to \widehat{\mathrm{H}}^{n-2d}(X_2, \Lambda(p-d)),$$

where Λ is $\mathbb{Q}$ or $\mathbb{R}$, and where $d = \dim X_1 - \dim X_2$. This is compatible with composition in the sense that given another map $g : X_2 \to X_3$ such that g and $g \circ f$ satisfy the assumptions, we have

$$g_* \circ f_* = (g \circ f)_*.$$

The assumptions apply in particular to the following two cases

- *X_1, X_2 regular and projective and f projective*

- *X_1 and X_2 smooth and quasi-projective over S, f proper: $i_1 : X_1 \to X_1' := X_1 \times_S X_2$ the closed immersion realizing the graph of f, $\pi_1 : X_1 \times_S X_2 \to X_2$ the projection, $X_2' = X_2$.*

Proof. The unit of the adjunction between f^* and f_* gives a map

$$\begin{aligned} p_{2!}{p_2}^!\mathbf{1}(d)[2d] &\to p_{2!}f_*f^*{p_2}^!\mathbf{1}(d)[2d] \\ &= p_{2!}f_*{i_1}^*{\pi_1}^*{i_2}^!{\pi_2}^!\mathbf{1}(d)[2d] \\ &= p_{2!}f_*{i_1}^*{\pi_1}^*{i_2}^*{\pi_2}^*\mathbf{1}(\dim X_1 - \dim S)[2(\dim X_1 - \dim S)] \\ &= p_{2!}f_!{i_1}^!{\pi_1}^!{i_2}^!{\pi_2}^!\mathbf{1} \\ &= p_{2!}f_!f^!{p_2}^!\mathbf{1} \\ &= p_{1!}{p_1}^!\mathbf{1} \end{aligned}$$

We have used the following things:

- Absolute purity for i_1, i_2: recall that this means that for any closed immersion i of codimension c between regular schemes, we have $i^!\mathbf{1} \cong i^*\mathbf{1}(-c)[-2c]$.

- Relative purity for π_1, π_2; relative purity means that for any smooth quasi-projective map π of relative dimension d, and any object M, we have $f^*(M)(d)[2d] \cong f^!(M)$

- The map f is proper, so $f_* = f_!$.

This gives the desired map on cohomology groups upon applying $\mathrm{Hom}_{\mathbf{DM_B}}(-, \widehat{\mathrm{H}})$. The compatibility statement follows since the absolute and relative purity isomorphisms are compatible with compositions. □

Remark 4.2.2. For comparison, the pushforward of arithmetic K-theory by Takeda is restricted to maps $f : X \to Y$ between arithmetic varieties (flat over $\mathbb{Z}$ and regular) that are proper and such that f is smooth projective. Moreover, the definition needs an auxiliary choice of a metric on the relative tangent bundle [63, Section 7.3.]. The pushforward on arithmetic Chow groups [30, Theorem 3.6.1] is defined for all proper maps between arithmetic varieties. For the time being, no pushforward has been established for the higher arithmetic Chow groups of Burgos and Feliu.

Chapter 5

Conjectures on zeta values

The content of this chapter is joint work in progress with Jakob Scholbach.

5.1 Relation to Scholbach's conjectural picture

In [55], Jakob Scholbach formulates a conjecture on special values of L-functions of Déglise-Cisinski motives over $\operatorname{Spec} \mathbb{Z}$, which can be seen as a reformulation of the Beilinson conjectures. The aim of this section is to give a brief review of this conjecture, and then discuss how Arakelov motivic cohomology fits into this picture.

5.1.1 Scholbach's conjecture

Recall from Section 2.2.1 the category $\mathbf{DM}_{\mathrm{B}}(\operatorname{Spec} \mathbb{Z})$, and its subcategory $\mathbf{DM}_{\mathrm{gm}}(\operatorname{Spec} \mathbb{Z})$ of compact objects (geometric motives in the terminology of Voevodsky). Scholbach formulates the following two conjectures. Parts of these conjectures only make sense under certain assumptions like existence of motivic t-structure, functional equation for motivic L-functions, etc. For expository purposes we suppress some technical details in the formulation. See [55] for the precise formulations, and [56] for a longer introduction to these ideas.

Conjecture 5.1.1 (Scholbach)**.** There is a family of functors

$$\mathrm{H}^i_{\mathrm{c}} : \mathbf{DM}_{\mathrm{gm}}(\operatorname{Spec}\mathbb{Z}) \to \mathbb{R}-mod \quad (i \in \mathbb{Z})$$

from the category of geometric motives to finite-dimensional $\mathbb{R}$-vector spaces, satisfying the following properties:

- Any distinguished triangle $M_1 \to M_2 \to M_3$ in $\mathbf{DM}_{\mathrm{gm}}(\operatorname{Spec}\mathbb{Z})$ gives rise to a long exact sequence

  $$\cdots \mathrm{H}^i_{\mathrm{c}}(M_1) \to \mathrm{H}^i_{\mathrm{c}}(M_2) \to \mathrm{H}^i_{\mathrm{c}}(M_3) \to \mathrm{H}^{i+1}_{\mathrm{c}}(M_1) \to \cdots$$

- For any M in $\mathbf{DM}_{\mathrm{gm}}(\operatorname{Spec}\mathbb{Z})$, there is a long exact sequence

  $$\cdots \mathrm{H}^i_{\mathrm{c}}(M) \to \mathrm{H}^i_{\mathrm{M}}(M)_{\mathbb{R}} \to \mathrm{H}^i_{\mathrm{w}}(M) \to \mathrm{H}^{i+1}_{\mathrm{c}}(M) \to \cdots$$

  involving the realization map (or Beilinson regulator) from motivic cohomology tensored with $\mathbb{R}$ to weak Hodge cohomology.

- For any M in $\mathbf{DM}_{\mathrm{gm}}(\operatorname{Spec}\mathbb{Z})$, there is a functorial and perfect pairing called the *global motivic duality pairing*

  $$\pi^i_{\mathrm{M}} : \mathrm{H}^i_{\mathrm{c}}(M) \times \mathrm{H}^{-i}_{\mathrm{M}}(DM)_{\mathbb{R}} \longrightarrow \mathbb{R}.$$

  Here DM is the Verdier dual of M, defined by $DM = \mathbf{RHom}(M, \mathbf{1}(-1)[-2])$.

- If the motive M comes from a motive of the form $h^{2m}(V, m)$ where V is a smooth projective variety over $\mathbb{Q}$ with a regular projective model X of absolute dimension d, then the pairing π^0_{M} agrees with the height pairing

  $$\mathrm{CH}^m(V)_{\mathbb{Q},\mathrm{hom}} \times \mathrm{CH}^{d-m}(V)_{\mathbb{Q},\mathrm{hom}} \to \mathbb{R}$$

  defined by Gillet-Soulé and Beilinson.

The second conjecture concerns special values of L-functions. Scholbach defines the L-functions of any geometric motive M over $\operatorname{Spec}\mathbb{Z}$ as

$$L(M,s) = \prod_p \left(\det(\mathrm{id} - \mathrm{Fr}^{-1}\, p^{-s} \mid (i_p^* M)_\ell) \right)^{-1}$$

the product taken over all finite primes. Here $i_p^* M$ is a motive over $\mathbb{F}_p$ and ℓ is some prime different from p for which we take ℓ-adic realization. We assume that standard conjectural properties of this expression holds true, like independence of ℓ, meromorphic continuation and functional equation. A feature of the formalism of motives over $\operatorname{Spec}\mathbb{Z}$ is that we never have to take inertia fixed points for this formula to make sense.

Conjecture 5.1.2 (Scholbach)**.** The order of vanishing of $L(M,s)$ at $s=0$ is given by

$$\operatorname{ord}_{s=0} L(M,s) = \sum_{i\in\mathbb{Z}} (-1)^{i+1} \dim \mathrm{H}^i_M(D(M)).$$

and the first nonvanishing Laurent coefficient at $s=0$, denoted $L^*(M,0)$, lies in

$$L^*(M,0) =_{\mathbb{Q}} \prod_{i\in\mathbb{Z}} \det(\pi^i_M)^{(-1)^{i+1}}.$$

up to a rational factor. Here D denotes the Verdier dual, and π_M is the pairing in the previous conjecture.

Compared to other formulations of the Beilinson conjectures in the literature, this has several nice features. First of all it gives a unified point of view on the central, near-central, and convergence points, in a statement which does not involve any categories of motives not known to exist. It also seems like the idea of thinking of the Beilinson conjectures in terms of duality is new, although Flach and possibly other people have also been thinking along similar lines. As Scholbach points out, the conjectural duality pairing has the same formal shape as global duality in arithmetic, and Poincare duality on an affine curve. Finally, the above notion of L-function unifies L-functions of varieties over $\mathbb{Q}$ (defined in terms of ℓ-adic cohomology), Hasse-Weil zeta functions of arithmetic schemes (defined in terms of counting elements in residue fields), and zeta functions of varieties over finite fields. If we assume a number of well-known "motivic" conjectures, the conjecture of Scholbach is essentially equivalent to the conjunction of the Beilinson conjectures on L-values [48], [54], Soulé's ICM conjecture on orders of vanishings of Hasse-Weil zeta functions [60], and Tate's conjecture on pole orders of zeta functions of varieties over finite fields [64].

Remark 5.1.3. Deligne cohomology agrees with weak Hodge cohomology at least for smooth projective varieties. Note that this is not the same as absolute Hodge cohomology (which would agree with a suitable truncation of Deligne cohomology). In particular the groups $\mathrm{H}^n_{\mathrm{D}}(V, \mathbb{R}(p))$ might be non-vanishing for $n > 2p$. Although this non-vanishing is referred to as pathological in [54, Remark, page 8], these are exactly the groups we want. The reason is that weak Hodge cohomology satisfies a certain duality which is not satisfied by absolute Hodge cohomology, which is important for Scholbach's conjecture to make sense, see [55, Section 1.2] for more details.

5.1.2 The role of Arakelov motivic cohomology

One of the main motivations for this thesis was to define a cohomology theory giving a reasonable candidate for the groups $\mathrm{H}_c(M)$ conjectured by Scholbach to exist. As part of current work in progress we are trying to prove that our definition of Arakelov motivic cohomology satisfies several of the properties required by the conjecture of Scholbach. I will sketch some of these ideas here.

In order to use Arakelov motivic cohomology in the setting of Scholbach's conjecture, we need a definition for all (geometric) motives over $\operatorname{Spec}\mathbb{Z}$ and not just a definition for arithmetic schemes. We make the following definition.

Definition 5.1.4. For any motive M in $\mathbf{DM}_{\mathrm{gm}}(\operatorname{Spec}\mathbb{Z})$, we define Arakelov motivic cohomology of M by

$$\widehat{\mathrm{H}}^i(M) := \operatorname{Hom}_{DM_{\mathrm{B}}(\operatorname{Spec}\mathbb{Z})}(M, \widehat{\mathrm{H}\mathbb{Q}}[i]).$$

As mentioned in the previous chapter, we must use the second definition of Arakelov motivic cohomology of a scheme if we require that the functor so defined on arithmetic schemes factors over the category of motives.

$\mathbb{Q}$-structure. For the last part of conjecture 5.1.2 we need the tensor product $\otimes_i \big(\det \widehat{\mathrm{H}}^i(M)_{\mathbb{R}}\big)^{(-1)^i}$ to be equipped with a $\mathbb{Q}$-structure. (This of course only makes sense if we assume that the product is finite and all vector spaces occurring are finite-dimensional). One naive candidate for such a $\mathbb{Q}$-structure comes simply from the fact the groups $\widehat{\mathrm{H}}^i(M)_{\mathbb{R}}$ are $\mathbb{Q}$-vector spaces tensored with $\mathbb{R}$. (Note here that although the $\mathbb{R}$-vector spaces are conjecturally finite-dimensional, the $\mathbb{Q}$-vector spaces are typically infinite-dimensional, because they sit in the long exact sequence together with Deligne cohomology groups which are $\mathbb{R}$-vector spaces). However, this is probably not be the right $\mathbb{Q}$-structure. Another candidate for a $\mathbb{Q}$-structure should come from the $\mathbb{Q}$-structures on motivic cohomology and Deligne cohomology, and the fact that there is a long exact sequence with these groups.

Duality pairing. We are not yet sure about how to think about the duality pairing. We would like to have a pairing between $\widehat{\mathrm{H}}^i(M)_{\mathbb{R}} = \mathrm{Hom}(M, \widehat{\mathrm{H}\mathbb{Q}}[i]) \otimes \mathbb{R}$ and $\mathrm{H}_M^{-i}(D(M))_{\mathbb{R}} = \mathrm{Hom}(D(M), \mathbf{1}[-i]) \otimes \mathbb{R}$ (all Hom groups in this paragraph are in $\mathbf{DM}_{\mathrm{B}}(\mathrm{Spec}\,\mathbb{Z})$). If we assume $D(D(M)) \simeq M$, i.e. that the motive M is dualizable, then we have

$$\begin{aligned}
\mathrm{Hom}(D(M), \mathbf{1}[-i]) &= \mathrm{Hom}(\mathbf{1} \otimes D(M), \mathbf{1}[-i]) \\
&= \mathrm{Hom}(\mathbf{1}, \mathbf{RHom}(D(M), \mathbf{1}[-i])) \\
&= \mathrm{Hom}(\mathbf{1}, \mathbf{RHom}(D(M)(-1)[i-2], \mathbf{1}(-1)[-2])) \\
&= \mathrm{Hom}(\mathbf{1}, D(D(M)(-1)[i-2]) \\
&= \mathrm{Hom}(\mathbf{1}, M(1)[2-i])
\end{aligned}$$

and we can define a pairing simply by composition:

$$\begin{aligned}
\mathrm{H}^i_{\mathrm{c}}(M) \times \mathrm{H}^{-i}_{\mathrm{M}}(DM)_{\mathbb{R}} &= \mathrm{Hom}(M, \widehat{\mathrm{H}\mathbb{Q}}[i]) \otimes \mathbb{R} \times \mathrm{Hom}(DM, \mathbf{1}[-i]) \otimes \mathbb{R} \\
&= \mathrm{Hom}(M(1)[2-i], \widehat{\mathrm{H}\mathbb{Q}}(1)[2]) \otimes \mathbb{R} \times \mathrm{Hom}(\mathbf{1}, M(1)[2-i]) \otimes \mathbb{R} \\
&\to \mathrm{Hom}(\mathbf{1}, \widehat{\mathrm{H}\mathbb{Q}}(1)[2]) \otimes \mathbb{R} \\
&= \widehat{\mathrm{H}}^2(\mathrm{Spec}\,\mathbb{Z}, \mathbb{R}(1)) \\
&\cong \mathbb{R}
\end{aligned}$$

(The last equality is Lemma 5.2.3 at the end of the next section). The problem now is that because we are over an arithmetic base, motives of smooth proper arithmetic schemes are dualizable (see [19, Prop 14.3.23]), but as far as I understand, a general motive is not. So this definition simply does not work in general. Maybe there is another way of defining a pairing between these groups, but it could also be that the pairing (and hence Scholbach's conjecture) only makes sense for motives which are dualizable. Or alternatively it could be that the correct phrasing of Scholbach's pairing should be different from the above, maybe in terms of motivic homology instead of motivic cohomology. One should also note that the above definition makes sense already on the level of $\mathbb{Q}$-vector spaces and gives a $\mathbb{Q}$-valued pairing there, which strongly suggests that the first candidate for a $\mathbb{Q}$-structure above is wrong, and the second is right, or otherwise any definition of the height pairing in this spirit would take values in $\mathbb{Q}$. We hope to come back to all this in our paper [34], and also to the precise relation between this pairing and the height pairing.

5.2 A recent idea of Soulé

In a recent letter to Bloch and Lichtenbaum [62], Soulé suggests the following new approach to special values of zeta functions of arithmetic schemes. For X a scheme of finite type over $\mathrm{Spec}\,\mathbb{Z}$, let $\zeta_X(s)$ be its Hasse-Weil zeta function:

$$\zeta_X(s) = \prod \frac{1}{1 - N(x)^{-s}}$$

(the product taken over all closed points x of X). Soulé considers a connected regular scheme X, flat and projective over $\mathbb{Z}$, of dimension d. For the motivic cohomology groups $\mathrm{H}^n_M(X, \mathbb{Z}(p))$, he assumes finite generation and vanishing outside the range $0 \leq n \leq 2p$. He also assumes existence of Arakelov motivic cohomology groups $\widehat{\mathrm{H}}^n(X, \mathbb{Z}(p))$ satisfying a long exact sequence like the one above [ref to above], but with a slightly different ending which probably corresponds to the use of a truncated version of Deligne cohomology. The groups $\widehat{\mathrm{H}}^n(X, \mathbb{Z}(p))$ are equipped with the strongest topology such that all the maps in the long exact sequence are continuous. Now Soulé considers the following modified groups

$$\widehat{\mathrm{H}}^n(X, \mathbb{Z}(p))^0 = \begin{cases} \widehat{\mathrm{H}}^n(X, \mathbb{Z}(p)) & \text{if } n < 2p \\ \overline{\mathrm{CH}}^p(X) & \text{if } n = 2p \text{ and } p < d \\ \ker(\widehat{\deg} : \overline{\mathrm{CH}}^p(X) \to \mathbb{R}) & \text{if } n = 2p \text{ and } p = d \end{cases}$$

Note that we expect the second line to be true by definition with our definition of Arakelov motivic cohomology so no modification should be necessary in this case. Soulé suggests that all these groups should be compact, and carry a natural Haar measure, possibly induced by a Haar measure on Deligne cohomology groups.

Conjecture 5.2.1 (Soulé)**.** The first nonvanishing Taylor coefficient of the zeta function is described by

$$\zeta^*_X(\dim(X) - p) = \pm \prod_n \mathrm{vol}(\widehat{\mathrm{H}}^n(X, \mathbb{Z}(p))^0)^{(-1)^n}$$

Example 5.2.2. We consider the basic example motivating Soulé's conjecture, to illustrate the conjecture itself but also the severe limitations in our current understanding of motivic cohomology with integral coefficients of arithmetic schemes. Let $X = \mathrm{Spec}\,\mathcal{O}_F$, where $\mathcal{O}_F$ is the ring of integers in some number field F. Write r_1 for the number of real places, r_2 for the number of complex places, h for the class number, μ for the group of roots of unity, r_{Dir} for the Dirichlet regulator map, and R_{Dir} for the covolume of the image of this map. Then the Deligne cohomology groups $\mathrm{H}^n_{\mathrm{D}}(X, \mathbb{R}(1))$

vanish for $n = 0$ and $n = 2$ (by the long exact sequence with de Rham and Betti cohomology), so the long exact sequence [ref here] for $p = 1$ reads

$$\begin{aligned} 0 \to \widehat{\mathrm{H}}^1(X, \mathbb{Z}(1)) \to \mathrm{H}^1_M(X, \mathbb{Z}(1)) &\xrightarrow{r_{Dir}} \mathrm{H}^1_{\mathrm{D}}(X, \mathbb{R}(1)) \\ &\to \widehat{\mathrm{H}}^2(X, \mathbb{Z}(1)) \to \mathrm{H}^2_M(X, \mathbb{Z}(1)) \to 0 \end{aligned}$$

In [34] we will prove that the regulator defined above agrees with the Beilinson regulator, which in the current example is just the Dirichlet regulator. Write Γ for the image of this regulator. One can show that the Deligne cohomology group in the middle of the sequence is a real vector space of rank $r_1 + r_2$ and it is well known that Γ spans a subspace of codimension 1. This implies that $\widehat{\mathrm{H}}^1(X, \mathbb{Z}(1)) = \mu$. The five-term exact sequence can be split into two short exact sequences, one being

$$0 \to \mu \to {\mathcal{O}_F}^* \to \Gamma \to 0$$

and the other being

$$0 \to \mathbb{R}^{r_1+r_2}/\Gamma \to \widehat{\mathrm{H}}^2(X, \mathbb{Z}(1)) \to \mathrm{H}^2_M(X, \mathbb{Z}(1)) \to 0$$

Now let's assume that $\mathrm{H}^2_M(X, \mathbb{Z}(1)) = \mathrm{Pic}(\mathcal{O}_F)$. Then $\widehat{\mathrm{H}}^2(X, \mathbb{Z}(1))$ sits in the same short exact sequence as $\widehat{\mathrm{Pic}}(X)$, and since the first group in this sequence is divisible, the two are isomorphic. Therefore, with the correct normalization of the Haar measure on the real vector space $\mathbb{R}^{r_1+r_2}$, we can interpret the class number formula

$$\zeta_F^*(0) = -\frac{\mathrm{vol}(\widehat{\mathrm{Pic}}^0(\mathcal{O}_F))}{\mathrm{vol}(\mu_F)} = -\frac{hR}{w}$$

as an instance of Soulé's conjecture. Here $\widehat{\mathrm{Pic}}^0$ denotes the kernel of the degree map on $\widehat{\mathrm{Pic}}$, and replacing $\widehat{\mathrm{Pic}}$ by this kernel has the effect of killing the noncompact part of $\mathbb{R}^{r_1+r_2}/\Gamma$.

Note that we assumed here that $\mathrm{H}^2_M(X, \mathbb{Z}(1)) = \mathrm{Pic}(\mathcal{O}_F)$ and that $\mathrm{H}^1_M(X, \mathbb{Z}(1)) = {\mathcal{O}_F}^*$. Apparently this is not known, not even in the case

$X = \operatorname{Spec}\mathbb{Z}$. In general, for any arithmetic scheme Y, there is a map from $\operatorname{Pic}(Y)$ to $\mathrm{H}^2_M(Y, \mathbb{Z}(1))$, see [19, 10.3.1], but when working over an arithmetic base, it not known that this is an isomorphism, although it is expected to be so. The lesson to take away from this example might be that if we insist on working both over an arithmetic base *and* with integral coefficients at the same time, we simply do not know enough about ordinary motivic cohomology to say anything meaningful at all about Arakelov motivic cohomology.

However, if we tensor everything with $\mathbb{R}$, as we do in the setting of Scholbach's conjecture, we can say something at least. In particular, we can prove the following basic fact which is needed in order to make sense of the definition of the pairing π_M above.

Lemma 5.2.3. *In the case $X = \operatorname{Spec}\mathbb{Z}$, we have $\widehat{\mathrm{H}}^2(X, \mathbb{R}(1)) \cong \mathbb{R}$.*

Proof. Whenever we work with real (or rational) coefficients and regular schemes, motivic cohomology groups can be described in terms of K-theory. In particular, we have $\mathrm{H}^2_M(\operatorname{Spec}\mathbb{Z}, 1)_{\mathbb{R}} = \operatorname{Gr}^1 K_0(\operatorname{Spec}\mathbb{Z})_{\mathbb{R}} = 0$. Now it is immediate from the above 5-term exact sequence that $\widehat{\mathrm{H}}^2(\operatorname{Spec}\mathbb{Z}, \mathbb{R}(1))$ is one-dimensional (and canonically identified with $\mathrm{H}^1_{\mathrm{D}}(X, \mathbb{R}(1))$ modulo the subspace spanned by the image of the regulator). □

Chapter 6

Products

The content of this chapter is joint work in progress with Peter Arndt.

6.1 Higher arithmetic intersection theory

When working with varieties over a field, the subject of intersection theory studies Chow groups and their product structure. In this setting, we also have Bloch's higher Chow groups, also equipped with a product structure, which is sometimes referred to as "higher" intersection theory. As explained in any introduction to Arakelov theory, the usual Chow theory does not give a good theory of intersection numbers for schemes of finite type over $\operatorname{Spec}\mathbb{Z}$, since the infinite prime is missing. One of the main features of the arithmetic Chow groups of Gillet and Soulé is that they are equipped with a product structure, which gives a good Arakelov-theoretic notion of intersection theory. In the work on higher arithmetic Chow groups of Burgos and Feliu [11], one of the key points is that they can equip their groups with a product structure, and hence obtain a notion of "higher arithmetic" intersection theory. As already mentioned, this works for varieties over a number field but not for arithmetic schemes. Having defined Arakelov motivic cohomology for arithmetic schemes, it is natural to ask if we can equip it with a product structure, and this is the question we address in this chapter.

We still lack an explicit cycle-theoretic definition of higher arithmetic Chow groups for arithmetic schemes, and it seems desirable to keep searching for such a definition, which should presumably agree with our groups for nice enough schemes. Assuming such a definition, together with functorial isomorphisms with Arakelov motivic cohomology groups, the product structure we define here would transfer to the higher arithmetic Chow groups, and give a notion of higher arithmetic intersection theory for arithmetic schemes.

6.2 Products on Arakelov motivic cohomology

Let us start by explaining the key idea. In order to get a product structure on the Arakelov motivic cohomology groups, we would like to equip the spectrum $\widehat{\mathrm{H}\mathbb{Z}}$ with the structure of a ring spectrum. Since this spectrum is the homotopy fiber of a map between ring spectra, our original idea was the following: Maybe there is a model structure on the category of (strict) ring spectra, such that taking the homotopy fiber *inside this model structure* gives something weakly equivalent to the homotopy fiber inside the category of spectra. This would give a ring spectrum structure on the Arakelov motivic cohomology spectrum.

There are several instances of model structures on commutative ring spectra in the literature, for example work of Shipley [59] in the setting of topological spectra, and work of Lurie on commutative monoids in stable infinity-categories [44, 4.3.21]. However, the problem with the above idea is that it turns out not to make sense, for the following reason. The homotopy fiber of a map $f : A \to B$ in a pointed model category is by definition the homotopy pullback of the diagram

$$\begin{array}{ccc} & & Z \\ & & \downarrow \\ A & \longrightarrow & B \end{array}$$

where the vertical map is the unique map from the zero object to B. However, this map is not a map in the category of ring spectra, because it does not preserve the multiplicative unit of the ring! (The zero object here is the zero ring, in which $0 = 1$). One can save the idea though, by simply forgetting about units, and instead work in the category of non-unital ring spectra. (This insight is due partially to conversations with Burt Totaro and Neil Strickland). In order for the homotopy fiber in non-unital ring spectra to be isomorphic to the usual homotopy fiber in **SH**, it would be enough to have model structures such that weak equivalences and fibrations are preserved by the forgetful functor from non-unital ring spectra to spectra.

Hence we are left with the following problem: Construct a model structure on non-unital motivic strict ring spectra such that weak equivalences (and fibrations) are also weak equivalences (and fibrations, respectively) in the underlying category of spectra. Our original approach to this problem was to mimick Shipley's construction [59], generalizing it to the motivic and non-unital setting. This seems to work, but while writing out the details of this, we learnt from Markus Spitzweck about some very recent results of Hornbostel which are much more general, and which make our first approach obsolete, at least for this particular application to Arakelov motivic cohomology. The work of Hornbostel is formulated in terms of model structures on categories of algebras over operads. In this thesis, we shall not review the theory of operads; all we need to know about operads is that the category of commutative monoid objects (unital or non-unital) in a given symmetric monoidal category $\mathcal{C}$ can be interpreted as the category of algebras over a certain operad.

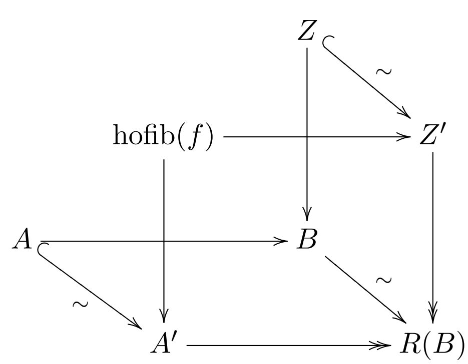
Theorem 6.2.1 (Hornbostel)**.** *Let* **Sp** *be the category of motivic symmetric spectra over a fixed base scheme S. Let* $\mathrm{Mon}(\mathbf{Sp})$ *be the category of strict ring spectra, i.e. the category of commutative monoid objects in* **Sp***, either unital or non-unital. Then there are model structures on* **Sp** *and* $\mathrm{Mon}(\mathbf{Sp})$ *such that the forgetful functor* $\mathrm{Mon}(\mathbf{Sp}) \to \mathbf{Sp}$ *sends fibrations to fibrations, and weak equivalences to weak equivalences, and such that the homotopy category of* **Sp** *is the usual motivic stable homotopy category* $\mathbf{SH}(S)$*.*

Proof. For the model structure on **Sp**, we can take Hornbostel's stable flat positive model structure [35, Thm 3.4]. For Mon(**Sp**) we can take the model structure of [35, Thm 3.6], where we take the operad $\mathcal{O}$ to be the commutative monoid operad, unital or non-unital. □

Theorem 6.2.2. *Let* **Sp** *be as in the previous theorem, and let* Mon(**Sp**) *now be the category of* unital *commutative monoid objects in* **Sp**. *Let* $f : A \to B$ *be a morphism in* Mon(**Sp**), *and let* H *be the homotopy fiber of* f. *Then* H *can be equipped with a non-unital strict ring spectrum structure, such that the map* $h : H \to A$ *in* **SH** *is a morphism of non-unital strict ring spectra.*

Remark 6.2.3. For the following proof to be valid, we actually have to assume that the model structures mentioned in Theorem 6.2.1 are proper, or otherwise the homotopy fiber defined as below may depend on choices made when choosing fibrant replacements. For the unital case, properness is established by Hornbostel [35, Thm 3.17], but he does not consider the non-unital case. We hope to come back to this point in a future publication.

Proof. We use double-headed arrows for fibrations, and curved arrow tails for cofibrations. Recall that in any pointed model category, the homotopy fiber of a map $f : A \to B$ can be computed as follows:



Here R denotes fibrant replacement, and the diagram means that we have replaced B by a fibrant replacement, then replaced the two maps in the diagram by fibrations (by factoring them into an acyclic cofibration followed by a fibration), and then taken pullback in the usual category-theoretic sense. Hornbostel's theorem tells us that using the right model structure on **Sp**, there is a model structure on non-unital monoids in **Sp** such that weak equivalences (and fibrations) are also weak equivalences (and fibrations, respectively) in the underlying model category **Sp**. Hence we can take all the fibrant replacements in the above diagram inside the category of non-unital monoids, and the resulting homotopy pullback will also be a homotopy pullback in **Sp**. (Here we also use that the forgetful functor from non-unital monoids to spectra commutes with pullbacks, which is a general fact about the forgetful functor from algebras over an operad to the underlying category, see [28, Prop. I.3.6].)

If like in the case of the spectrum $\widehat{\mathrm{H}\mathbb{Z}}$, the map f is represented by a zig-zag rather than an actual map in the model category, one has to first replace the domain and target of f, cofibrantly and fibrantly respectively. After such replacements, the map in the homotopy category lifts to an actual map in the model category. For the purposes of getting a non-unital monoid structure on the homotopy fiber, we take these replacements inside the model category of unital monoids, and then take the homotopy fiber as above inside the model category of non-unital monoids. □

As a consequence of the above theorem, we get the following corollary. Recall that complexes computing weak Hodge cohomology, with the properties required by this corollary, do exist according to private communication with Burgos, but we have not yet checked the details of this.

Corollary 6.2.4. *Assume that we have Deligne complexes E_p as in Section 3.2, but with products that are strictly commutative and associative. Let $\mathcal{E}_{\mathrm{D}}$ be the associated Deligne spectrum. Then the spectrum $\widehat{\mathrm{H}\mathbb{Z}}$, defined in Definition 3.3.3, has a structure of non-unital strict ring spectrum, and the map $\widehat{\mathrm{H}\mathbb{Z}} \to \mathrm{H}_{\mathcal{E}}$ is a map of non-unital strict ring spectra, so in particular it induces a map on cohomology groups which respects products.*

Remark 6.2.5. It is interesting that the above argument would not work if we wanted to take a homotopy cofiber instead of a homotopy fiber.

Remark 6.2.6. Recall that motivated by Scholbach's conjecture, we expect the groups $\widehat{\mathrm{H}}^{2p}(X, \mathbb{Z}(p))$ to agree with $\overline{\mathrm{CH}}^p(X)$ for sufficiently nice arithmetic schemes X. The idea of the spectrum $\widehat{\mathrm{H}\mathbb{Z}}$ being a non-unital ring spectrum fits well with the fact that $\overline{\mathrm{CH}}^*(X)$ is a non-unital ring (it is a proper ideal of the unital ring $\widehat{\mathrm{CH}}^*(X)$).

Chapter 7

Gluing spectra over $\operatorname{Spec}\mathbb{Z}$

7.1 Introduction

In this chapter, which is independent from the rest of the thesis, we prove a gluing result for objects in $\mathbf{SH}(\operatorname{Spec}\mathbb{Z})$, i.e. for spectra representing cohomology theories for arithmetic schemes. Much of this material has been influenced by conversations with Joseph Ayoub.

The basic question of this chapter is the following: Given a spectrum E in $\mathbf{SH}(\operatorname{Spec}\mathbb{Z})$, we can pull it back to all prime fields, i.e. to $\operatorname{Spec}\mathbb{Q}$ and to $\operatorname{Spec}\mathbb{F}_p$ for any finite prime p. A natural question is: Do these pullbacks determine E uniquely? One could also ask the following: Given a spectrum over $\operatorname{Spec}\mathbb{Q}$, and spectra over each $\operatorname{Spec}\mathbb{F}_p$, does there exist a spectrum over $\operatorname{Spec}\mathbb{Z}$ with precisely these spectra as pullbacks? Good answers to these and related questions could in principle reduce certain problems related to cohomology theories for arithmetic schemes to the study of cohomology theories for varieties over fields, which are better understood.

The gluing results proved below give some kind of answer to the two questions above. It turns out that there is a kind of gluing formalism similar to the ones that exist for example for perverse sheaves or etale sheaves. More precisely, a spectrum E as above is determined by the pullbacks together with additional "gluing maps". Conversely, given spectra over each prime field together with gluing maps, one can construct a spectrum over $\operatorname{Spec}\mathbb{Z}$.

My original motivation for asking these questions was that I thought I could construct the Arakelov motivic cohomology spectrum over $\operatorname{Spec}\mathbb{Z}$ by specifying a spectrum over each prime field together with suitable gluing maps. However, this turned out to be a blind alley, and I later realized that the method used in Chapter 3 was much better. However, maybe this kind of gluing could have other applications in the future.

In this chapter, we shall use the terms homotopy colimits and homotopy fibers/cofibers in the sense used in the theory of triangulated categories. In particular, a homotopy fiber or cofiber of a map f is defined by sitting in a distinguished triangle together with f, and is thus non-functorial and unique only up to a non-canonical isomorphism. Also, a homotopy colimit of a diagram indexed by the natural numbers

$$X_0 \xrightarrow{a_0} X_1 \xrightarrow{a_1} X_2 \xrightarrow{a_2} \dots$$

can be defined up to non-canonical isomorphism by the following distinguished triangle

$$\oplus X_n \to \oplus X_n \to \operatorname{hocolim}_n X_n \to \oplus X_n[1]$$

where the first map is defined by a_i on the X_i summand (for all i).

7.2 The gluing statements

For comparison, let us start by recalling the following classical gluing theorem for etale sheaves of abelian groups:

Theorem 7.2.1. *[1, Cor 2.5] Write $Sh_{et}(Y)$ for the category of etale sheaves of abelian groups on a (separated) scheme Y. Consider a closed subscheme $i : Z \to X$ with open complement $j : U \to X$. Then the category $Sh_{et}(X)$ is equivalent to the category of triples (B, A, φ), where B is an object of $Sh_{et}(U)$, A is an object of $Sh_{et}(Z)$, and $\varphi : A \to i^*j_*B$ is a morphism of sheaves.*

Now we turn to the setting of spectra over $\operatorname{Spec}\mathbb{Z}$. Write $i_p : \operatorname{Spec}\mathbb{F}_p \to$

Spec$\mathbb{Z}$ for the inclusion of a closed point, and $\eta : \mathrm{Spec}\,\mathbb{Q} \to \mathrm{Spec}\,\mathbb{Z}$ for the generic fiber. Because i_p is a closed immersion, we can identify i_{p*} and $i_{p!}$ and will do so throughout the chapter. Note that given a spectrum E in $\mathbf{SH}(\mathrm{Spec}\,\mathbb{Z})$, we can consider all the pullbacks η^*E and i_p^*E, and for each p, there is a natural map from i_p^*E to $i_p^*\eta_*(\eta^*E)$ given by applying i_p^* to the unit map $id \to \eta_*\eta^*$. This observation together with the above-mentioned theorem gives some motivation for the following gluing construction.

Gluing construction. We begin with the following collection of "gluing data":

- For each prime p, a spectrum E_p over $\mathbb{F}_p$.
- A spectrum E_η over $\mathbb{Q}$.
- For each prime p, a map of spectra $\varphi_p : E_p \to i_p^*\eta_*E_\eta$, such that the collection $\{\varphi_p\}$ satisfies condition (A) described below.

We produce a spectrum over $\mathbb{Z}$ as follows. For each p, let K_p be the homotopy cofiber of φ_p:

$$E_p \to i_p^*\eta_*E_\eta \to K_p \to E_p[1] \tag{7.1}$$

For each p, let ψ_p be the composition

$$\eta_*E_\eta \to i_{p*}i_p^*\eta_*E_\eta \to i_{p*}K_p \tag{7.2}$$

where the first map is the unit of the i_p-adjunction at η_*E_η and the second one is i_{p*} applied to the second map of (7.1). Now take the product over all primes of the maps ψ_p, and assume that the resulting map $\eta_*E_\eta \to \prod i_{p*}K_p$ factors through the direct sum (this is condition (A)). Define E to be the homotopy fiber of this map to the direct sum:

$$E \to \eta_*E_\eta \to \oplus_p i_{p*}K_p \to E[1] \tag{7.3}$$

This E is a spectrum over Spec$\mathbb{Z}$. We say that E is obtained from the gluing construction applied to the gluing data above.

Proposition 7.2.2. *For any spectrum F over* $\operatorname{Spec}\mathbb{Z}$*, the following is a distinguished triangle in* $\mathbf{SH}(\operatorname{Spec}\mathbb{Z})$*:*

$$\oplus\, i_{q!}i_q^!F \to F \to \eta_*\eta^*F \to \oplus i_{q!}i_q^!F[1] \tag{7.4}$$

where the maps are given by counit and unit of the adjunctions in question.

Proof. For a closed subscheme $i : Z \to \operatorname{Spec}\mathbb{Z}$ with complement $j : U \to \operatorname{Spec}\mathbb{Z}$, we know that

$$i_*i^!F \to F \to j_*j^*F \to i_*i^!F[1]$$

is a distinguished triangle (this a special case of [2, Prop 1.4.5]). First we note that the triangle in the proposition is the homotopy colimit of triangles of this form. To see this for the first item in the triangle, we use the fact that a countable direct sum in any triangulated category is a homotopy colimit. This follows from the fact that the direct sum of distinguished triangles is distinguished [47][Remark 1.2.2], together with the above-mentioned description of (sequential and countable) homotopy colimits in any triangulated category. For the second item, we use that F is always the homotopy colimit of a diagram in which all maps are the identity on F, see [47, Lemma 1.6.6](ref Neeman, Lemma 1.6.6 page 64). For the third item, we refer to [18]. Now a distinguished triangle in **SH** is by definition a cofiber sequence, thus in particular a homotopy colimit diagram. But since homotopy colimits commute with homotopy colimits [16, 24.5], any homotopy colimit of distinguished triangles is also distinguished, and this applies in particular to the triangle in the proposition. □

Proposition 7.2.3. *Let $(E_\eta, \{E_p\}, \{\varphi_p\})$ be a collection of gluing data as above, and let E be the spectrum given by the gluing construction. Then for each p, E_p and i_p^*E are isomorphic in $SH(\mathbb{F}_p)$ and similarly for E_η and η^*E in $SH(\mathbb{Q})$.*

Proof. Apply i_p^* to the distinguished triangle (7.3) which defines E. This gives the following distinguished triangle in $\mathbf{SH}(\mathrm{Spec}\,\mathbb{F}_p)$:

$$i_p^*E \to i_p^*\eta_*E_\eta \to i_p^* \oplus_q i_{q*}K_q \to i_p^*E[1]$$

where the third item can be identified with K_p, using $i_p^*i_{p*} = id$ and $i_p^*i_{q*} = 0$ for $p \neq q$, and also that upper star functors are left adjoints so preserve colimits and in particular commute with direct sums. Hence the spectra i_p^*E and E_p both sit as fibers in the same fiber sequence, so they must be (non-canonically) isomorphic in $\mathbf{SH}(\mathbb{F}_p)$.

Now apply η^* to 7.3. Using that $\eta^*\eta_* = id$ and $\eta^*i_{p*} = 0$, we get a distinguished triangle in $\mathbf{SH}(\mathrm{Spec}\,\mathbb{Q})$:

$$\eta^*E \to E_\eta \to 0 \to \eta^*E[1]$$

which completes the proof. □

Proposition 7.2.4. *Let F be a spectrum over $\mathbb{Z}$. Let $(E_\eta, \{E_p\}, \{\varphi_p\})$ be the gluing data obtained from F by pullbacks. This gluing data satisfies condition (A) in the gluing construction. Furthermore, the spectrum E obtained by applying the gluing construction to this gluing data is isomorphic to the spectrum F we started with.*

Proof. Apply i_p^* to (7.4). This gives

$$i_p^!F \to i_p^*F \to i_p^*\eta_*\eta^*F \to i_p^!F[1]$$

and since K_p is by defininition the cofiber of the second map in this sequence, we get

$$K_p[-1] \simeq i_p^!F$$

(non-canonical isomorphism in $SH(\mathbb{F}_p)$). Hence using Proposition 7.2.2 there is a distinguished triangle

$$F \to \eta_*\eta^*F \to \oplus i_{q*}K_q \to F[1] \tag{7.5}$$

so in particular we have a map from $\eta_*\eta^*E$ to $\oplus i_{q*}K_q$, call it g. To show that condition (A) is satisfied, it is enough to show that the original map in the gluing construction (i.e. the product of all ψ_p) factors through g. But this follows by inspection of the definitions of g and ψ_p, after observing that both arise via a construction involving the cofiber of the unit $id \to \eta_* \eta^*$.

Furthermore, E is defined by sitting in the distinguished triangle

$$E \to \eta_* \eta^* F \to \oplus i_{q*} K_q \to E[1]$$

and comparing this to the sequence 7.5 shows that E and F are (non-canonically) isomorphic. □

7.3 An equivalence of categories?

For the purposes of constructing a spectrum over $\operatorname{Spec} \mathbb{Z}$ with prescribed pullbacks to each prime field, the above results are sufficient, provided one can find a way of constructing gluing maps. However, it is natural to ask whether one can refine the above results into an equivalence of categories. The idea would be the following. There is an obvious notion of morphism of gluing data which makes the class of all collections of gluing data into a category, call it $\mathcal{G}$. The hope would be that there is an equivalence of categories between $\mathbf{SH}(\operatorname{Spec} \mathbb{Z})$ and $\mathcal{G}$. The process described above which constructs a collection of gluing data from a spectrum E, defines a functor B from $\mathbf{SH}(\operatorname{Spec} \mathbb{Z})$ to $\mathcal{G}$, and it should be easy to show that it is faithful. However, using only the methods above, there is (as far as I can see) no obvious way of proving that this functor is full, or essentially surjective, or that there is a functor in the other direction. A main obstacle is that cones are not functorial in a triangulated category.

There are at least two possible ways of getting around this problem. Firstly, one could try to work in a context where cones are functorial. This might be enough for proving that there is an equivalence of categories as desired. I'm not sure what this context would be, but I imagine that there is a notion of simplicial enhancement of triangulated categories which gives a setting in which cones are actually functorial, analogous to the DG enhancement described in [8]. Or maybe one could try to work with a fixed choice of functorial fibrant/cofibrant replacement functors on the level of the underlying model categories. However, another approach might be to instead do everything on the level of model categories. By this I mean that we can consider a notion of gluing data based on the model category underlying **SH**, and the functor B above will lift to this level. Then the hope would be that there is a natural model structure on the category of such gluing data, such that passing to the homotopy category gives back $\mathcal{G}$, and such that (the lift of) B is part of a Quillen equivalence. The advantage of this approach compared to the first is that one could then say that strict ring spectra in $\mathbf{SH}(\mathrm{Spec}\,\mathbb{Z})$ correspond to monoids in the model category underlying $\mathcal{G}$, so questions about such ring spectra and their categories of modules could be reduced to similar questions over base schemes which are fields. We remark here that the upper and lower star functors for **SH** lift to the level of model categories, and are monoidal and lax monoidal respectively, so send monoids to monoids (otherwise the hoped-for application to strict ring spectra would not make sense).

Chapter 8

Work in progress and future directions

In this final chapter we describe some work in progress, and list a few natural open questions.

8.1 Properties of Arakelov motivic cohomology

8.1.1 Comparison with the Beilinson regulator

In [34], we will prove together with Scholbach that the map from motivic cohomology to Deligne cohomology defined by the regulator of Corollary 3.3.2 agrees with the classical Beilinson regulator. This proof is based on techniques from Riou's thesis.

8.1.2 Comparison with arithmetic Chow groups

We have defined the groups $\widehat{\mathrm{H}}^n(X, p)$ for X an arithmetic scheme as well as a variety over $\mathbb{Q}$. As part of current work in progress we are investigating how these groups compare with the Gillet-Soulé arithmetic Chow groups. In the case of varieties over $\mathbb{Q}$ we also want to make the comparison with the Burgos-Feliu higher arithmetic Chow groups. We conjecture that the following holds. Firstly, if X is a regular projective arithmetic scheme or a smooth projective variety over $\mathbb{Q}$, we expect that

$$\widehat{\mathrm{H}}^{2p}(X,p) \cong \overline{\mathrm{CH}}^p(X)$$

where the left hand side is given by the definition in Chapter 4 and the right hand side is the Gillet-Soulé groups defined in Section 2.4.1. For the "higher" groups, we expect that when $2p > n$, for any smooth variety X over $\mathbb{Q}$ we should have

$$\widehat{\mathrm{H}}^n(X,p) \cong \widehat{\mathrm{CH}}^p(X, 2p-n)$$

where the right hand side is given by the higher Chow groups of Burgos and Feliu.

Gillet has asked in a talk whether it is possible to express the usual arithmetic Chow groups $\widehat{\mathrm{CH}}(X)$ as hypercohomology groups of a sheaf of DGAs. According to notes from this talk, the approach suggested by him would rely on unproven cases of the Gersten conjecture, and also on some difficult analytic work at the infinite prime. Together with the expected comparison just mentioned, the results in this thesis can be viewed as a partial affirmative answer to Gillet's question, using completely different tools. The difference is that we can only treat the subgroup $\overline{\mathrm{CH}}(X) \subset \widehat{\mathrm{CH}}(X)$ and that we express this group as cohomology represented by a (nonunital) motivic ring spectrum rather than hypercohomology of a sheaf of DGAs.

8.2 Computing examples

It would be interesting to try to compute some Arakelov motivic cohomology groups for some simple but nontrivial schemes. It seems like the only cases where one could hope to compute much is where we know something about the regulator map. I hope to be able to look at some specific curves in the future, most likely some modular, elliptic, Fermat or hyperelliptic curves.

8.3 Weil-etale topology

In [41], Lichtenbaum introduced the idea of Weil-etale topology, with the purpose of formulating conjectures on the precise special values of zeta functions of arithmetic schemes over $\mathrm{Spec}(\mathbb{Z})$. These ideas have been developed much further by Flach and Morin [27], who are able to formulate precise conjectures and prove some results in the $s = 0$ case. For the study of zeta functions at non-zero integers, recent work in progress of Flach indicate that some version of Arakelov motivic cohomology should enter the picture in a crucial way, and that a pairing similar to the one in Scholbach's conjecture also plays a role. However, it seems like the groups defined in this thesis are not exactly right for the setting of Flach. Several modifications might be necessary, one being that one should use Deligne cohomology with integral coefficients instead of real coefficients. To construct a regulator from motivic cohomology to integral coefficients Deligne cohomology, one must use some method different from the one used in this thesis, since Theorem 3.3.1 does not apply. One possible route to such a definition might go via the work of Lima-Filho [42].

8.4 Higher arithmetic Riemann-Roch theorems

The classical Grothendieck-Riemann-Roch theorem relates the pushforward functoriality on algebraic K_0 to that on Chow groups. One version of this theorem can be formulated as follows [29]. Let S be a Dedekind domain, and let Y and X be regular schemes, quasi-projective and flat over S. Let $g : Y \to X$ be a flat and projective S-morphism. Then the following diagram is commutative:

$$\begin{array}{ccc} K_0(Y) & \xrightarrow{Td(g)\cdot ch} & \mathrm{CH}^*(Y)_{\mathbb{Q}} \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} \\ K_0(X) & \xrightarrow{ch} & \mathrm{CH}^*(X)_{\mathbb{Q}} \end{array}$$

Here ch is the Chern character, and $Td(g)$ is the so called Todd class of the virtual relative tangent bundle. This commutative square has been generalized in two directions. Firstly, a *higher* version can be formulated, where the left hand side of the diagram is replaced by higher algebraic K-groups and the right hand side by higher Chow groups or motivic cohomology groups. Secondly, there is an *arithmetic* generalization involving arithmetic $\widehat{\mathrm{K}}_0$ and arithmetic Chow groups. Various people have done work on versions of the arithmetic Riemann-Roch square, including Gillet and Soulé, Faltings, Rössler, Zha, and Burgos. See [29] and [14] for the most recent results and a brief survey.

Now a natural question is whether it is possible to formulate a higher arithmetic Riemann-Roch theorem, involving higher arithmetic K-groups on the left hand side. The question then is what groups should occur on the right hand side. We hope that the Arakelov motivic cohomology groups will fit into this picture, leading to a square of the form

$$\begin{array}{ccc} \widehat{\mathrm{K}}_n(Y) & \xrightarrow{(??)\cdot ch} & \prod_i \widehat{\mathrm{H}}^{2i-n}(Y, \mathbb{Q}(i)) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ \widehat{\mathrm{K}}_n(X) & \xrightarrow{ch} & \prod_i \widehat{\mathrm{H}}^{2i-n}(X, \mathbb{Q}(i)) \end{array}$$

To arrive at a theorem like this, a lot of work remains to be done. At the moment it is not at all clear what the correction term (some kind of higher arithmetic Todd class?) should look like. It is also not clear what class of schemes one can expect this to work for, or what conditions one should impose on g in addition to being proper. There is also a choice to make between the various different versions of higher arithmetic K-groups, due to Deligne, Soulé and Takeda. Finally, we need to define higher Chern class maps, but this is something we are currently in the process of doing.

8.5 Hodge realizations

Given a strict ring spectrum, it is natural to study the category of modules over it. Sometimes this category is equivalent to some other interesting category obtained in a different way. For example, at least when the base scheme is a field of characteristic zero, the homotopy category of the category of modules over the motivic cohomology spectrum is equivalent to Voevodsky's triangulated category **DM** of motives [53]. Other examples arises from spectra representing mixed Weil cohomologies [20]. In this setting, writing $\mathcal{E}$ for the representing spectrum, tensoring with $\mathcal{E}$ induces a functor from the category of Beilinson motives to the homotopy category of $\mathcal{E}$-modules. The latter can be identified with the derived category of $\mathbf{K}$-vector spaces, where $\mathbf{K}$ is the coefficient field of the Weil cohomology in question [19, Cor 16.2.11].

Let $\mathcal{E}_{\mathrm{D}}$ be a strict ring spectrum representing some version of Deligne cohomology or Hodge cohomology. One may ask whether it is possible to describe the homotopy category $\mathbf{Ho}(\mathcal{E}_{\mathrm{D}} - Mod)$ in some way. According to the general philosophy of Cisinski and Déglise [19, Chapter 16], this category should be the target for a realization functor from Beilinson motives to $\mathbf{Ho}(\mathcal{E}_{\mathrm{D}} - Mod)$, and therefore a natural guess is that $\mathbf{Ho}(\mathcal{E}_{\mathrm{D}} - Mod)$ is equivalent to the derived category of some suitable abelian category of mixed Hodge structures, and the functor given by tensoring with $\mathcal{E}$ should be identified with the corresponding Hodge realization functor. According to Cisinski [18], any realization functor should have a right adjoint, and the Deligne spectrum should be the image of the unit object under the right adjoint of the weak Hodge realization functor. I do not know how to prove any of these things.

8.6 Alternative forms of motivic homotopy theory?

The framework of motivic homotopy theory works very well when the base scheme is a field, but for some arithmetic applications, it might be necessary to develop other forms of homotopy theory. Motivic homotopy theory is based on simplicial presheaves (or sheaves) on the category of smooth schemes over a base scheme S. One could imagine working with simplicial presheaves on some other category. Some natural candidates would be:

- The category of regular schemes (satisfying some finiteness or cardinality condition). Some remarks on this idea are given by Riou, see [52, Remark 4.3.3].

- The category of Arakelov varieties in the sense of Gillet and Soulé [30, Section 5.1], i.e. arithmetic schemes together with a choice of "metric at infinity".

- Some category of "generalized schemes" containing a compactification $\overline{\operatorname{Spec}\mathbb{Z}}$ of $\operatorname{Spec}\mathbb{Z}$ in some sense, and maybe some kind of schemes over some notion of the field with one element. Many such categories have been suggested in recent years, by Durov, Haran, Soulé, Toen and others; see [43] for a survey.

Another useful modification of motivic homotopy theory might be to not require $\mathbb{A}^1$-invariance. Maybe this would make it possible to express non-$\mathbb{A}^1$-homotopy invariant cohomology theories (like the full arithmetic Chow groups of Gillet and Soulé) as representable functors. Another reason for looking in this direction is that in the Weil-etale framework already mentioned, one might want to consider regulators from etale motivic cohomology. When doing so there are problems with p-torsion which might be more easily handled with non-$\mathbb{A}^1$-invariant machinery. It is worth noting that Toen's theory of schematic homotopy types does give a kind of homotopy theory for schemes in which $\mathbb{A}^1$-invariance is not required. However, it is not clear if this is of direct relevance to arithmetic applications.

8.7 Arakelov motives?

One may ask if there it is possible to define a reasonable notion of "Arakelov motives". Although it is not completely clear what this should mean, one reasonable thing to do would be to construct a triangulated category similar to Voevodsky's triangulated category of motives, but such that Hom groups are Arakelov motivic cohomology groups rather than motivic cohomology groups. There are several possible approaches to such a construction:

- One could try to mimick Voevodsky's original construction, but using arithmetic correspondences in the sense of [32] rather than usual correspondences. As a first step one would have to understand what the analogue of finite correspondences should be.

- Another approach would be to start with some category of "generalized schemes" as above, and work with simplicial sheaves/presheaves on this category. Maybe one could then define $\mathbf{SH}(\overline{\operatorname{Spec}\mathbb{Z}})$ and obtain a definition of $\mathbf{DM}(\overline{\operatorname{Spec}\mathbb{Z}})$ either as a subcategory of this $\mathbf{SH}$ category or as some category of modules.

- Finally, the simplest approach would be to take the spectrum $\widehat{\mathrm{H}\mathbb{Z}}$ defined in this thesis, and take the category of Arakelov motives to be the homotopy category of $\widehat{\mathrm{H}\mathbb{Z}}$-modules. If $\widehat{\mathrm{H}\mathbb{Z}}$ had been a unital ring spectrum, this would give a triangulated category in which the Hom groups are Arakelov motivic cohomology groups as desired (in the sense of Chapter 3), simply because tensoring with $\widehat{\mathrm{H}\mathbb{Z}}$ is Quillen left adjoint to the forgetful functor from $\widehat{\mathrm{H}\mathbb{Z}}$-modules to spectra. However, because the spectrum is non-unital, one has to be more careful. One could either try to verify the same adjunction for non-unital ring spectra, or one could try to make precise the notion of adjoining a unit to a ring spectrum.

Going further, one could hope for a good t-structure on one of these triangulated category of "Arakelov motives". There are two things one could maybe wish for with regards to the heart of such a t-structure.

- It should be possible to construct a category of "pure" Arakelov motives using arithmetic correspondences, analogous to the classical construction of pure Chow motives [57], and one could hope that this category is closely related to the heart of the hoped-for t-structure.

- Many cohomology theories come in pairs of an "absolute" and a "geometric" theory, see [48, Section 3] for the general pattern of such pairs. The geometric cohomology typically takes values in some highly structured Tannakian category, and the absolute cohomology groups arise as Ext groups in this Tannakian category. For example, various versions of Hodge cohomology arise as Ext groups in a suitable category of Hodge structures, and motivic cohomology (also an absolute cohomology theory) conjecturally arises as Ext groups in an abelian (Tannakian) category of mixed motives. Now one can ask if Arakelov motivic cohomology fits in such a pair together with some yet unknown geometric cohomology theory. A very optimistic guess would be that such a geometric theory is closely related to Deninger's conjectural cohomology theory [22], sometimes referred to as archimedean cohomology.

Appendix A

The equivariance problem

A.1 Introduction

Recall from section 2.2.1 the following definition. Let $\mathcal{C}$ be a symmetric monoidal category and let T be an object of $\mathcal{C}$. A symmetric T-spectrum in $\mathcal{C}$ is a sequence of objects $\{F_n\}_{n=0}^{\infty}$ together with actions of the symmetric group S_n on F_n and "bonding" morphisms $T \otimes F_n \to F_{n+1}$ such that the induced maps $T^{\otimes m} \otimes F_n \to F_{n+m}$ are $S_m \times S_n$-equivariant. A morphism of symmetric spectra from $F = \{F_n\}_{n=0}^{\infty}$ to $F' = \{F'_n\}_{n=0}^{\infty}$ is a sequence of morphisms $a_n : F_n \to F'_n$ which are S_n-equivariant and commute with the bonding maps.

In chapter 2 of thesis we described the construction of the category $\mathrm{D}_{\mathbb{A}^1,\mathbb{Q}}$ in terms of symmetric $\mathbb{Q}(1)$-spectra, and we gave a definition of the Deligne spectrum in this setting. However, as pointed out by the examiners, it is not clear from the definition that the symmetric group actions (which were defined to be trivial) really satisfy the equivariance requirement above. In this appendix we fix this problem by giving a new definition of the Deligne spectrum, but this time as a $\mathbb{P}^1$-spectrum. For this, we have to work in **SH** rather than in $\mathrm{D}_{\mathbb{A}^1,\mathbb{Q}}$, and we have to use a construction of **SH** which is slightly different from the one described in section 2.2.1, following Jardine [37].

A.2 Preliminaries

A.2.1 The stable homotopy category

Let S be a Noetherian scheme. Unless explicitly mentioned otherwise, all morphisms of schemes are understood to be separated and of finite type. Recall that $\mathbf{Sm}/S$ is the category of smooth schemes over S. The category of presheaves of pointed sets on this category is denoted by $\mathbf{PSh}_\bullet := \mathbf{PSh}_\bullet(\mathbf{Sm}/S)$. We identify a scheme $X \in \mathbf{Sm}/S$ with the presheaf (of sets) represented by X, and we write $X_+ := X \sqcup \{*\}$ for this presheaf with a disjoint basepoint added. We will view the projective line $\mathbb{P}^1_S$ as pointed by ∞. The prefix Δ^{op} indicates simplicial objects in a category. The simplicial n-sphere is denoted S^n, this should not cause confusion with the base scheme S. We consider the following Quillen adjunctions between model categories, where we have introduced notation for the homotopy category of each model category underneath.

$$\begin{array}{ccccccccc} & \mathrm{id} & & \mathrm{id} & & \Omega^\infty & & \mathrm{id} & \\ \Delta^{\mathrm{op}}\mathbf{PSh}_\bullet & \leftrightarrows & \Delta^{\mathrm{op}}\mathbf{PSh}_\bullet & \leftrightarrows & \Delta^{\mathrm{op}}\mathbf{PSh}_\bullet & \leftrightarrows & \mathbf{Sp}^{\mathbb{P}^1}_\bullet & \leftrightarrows & \mathbf{Sp}^{\mathbb{P}^1}_\bullet \\ & \mathrm{id} & & \mathrm{id} & & \Sigma^\infty_{\mathbb{P}^1} & & \mathrm{id} & \\ \mathbf{Ho}_{sect,\bullet} & & \mathbf{Ho}_{Nis,\bullet} & & \mathbf{Ho}_\bullet & & \mathbf{SH}_p & & \mathbf{SH} \end{array} \tag{A.1}$$

From left to right, the involved model structures are the following (recall the 2-out-of-3-principle, by which any two classes among fibrations, cofibrations and weak equivalences determine the third): the *sectionwise model structure* on the category of pointed simplicial presheaves on smooth schemes over S is defined by sectionwise weak equivalences and monomorphisms as cofibrations. Second, the *Nisnevich local model structure* is determined by weak equivalences on Nisnevich stalks and monomorphisms as cofibrations. The $\mathbb{A}^1$-*local model structure* on presheaves is given by $\mathbb{A}^1$-equivalences and monomorphisms.

The next category, $\mathbf{Sp}^{\mathbb{P}^1}(\Delta^{\mathrm{op}}\mathbf{PSh}_\bullet(\mathbf{Sm}/S))$, consists of symmetric $\mathbb{P}^1_S$-spectra. It is endowed with the *projective model structure* [37, 2.1]: weak equivalences (fibrations) are maps $E \to F$ such that each $E_n \to F_n$ is an $\mathbb{A}^1$-equivalence (fibration) in the $\mathbb{A}^1$-local model structure. Moreover, cofibrations are maps such that $E_0 \to F_0$ is a cofibration and

$$\mathbb{P}^1 \wedge F_n \wedge_{\mathbb{P}^1 \wedge E_n} E_{n+1} \to F_{n+1}$$

is a cofibration. The functor

$$\Sigma^\infty_{\mathbb{P}^1} : \Delta^{\mathrm{op}}(\mathbf{PSh}_\bullet) \ni F \mapsto \big((\mathbb{P}^1)^{\wedge n} \wedge F\big)_{n \geq 0}$$

(bonding maps are identity maps, S_n acts by permuting the factors $\mathbb{P}^1$) is left adjoint to $\Omega^\infty : (E_n) \mapsto E_0$. Often, we will not distinguish between a simplicial presheaf F and $\Sigma^\infty_{\mathbb{P}^1}(F)$. Finally, the *stable model structure* is defined by projective cofibrations (the same as in the previous step) and stable $\mathbb{A}^1$-equivalences. The latter are defined as follows: an Ω-spectrum is an object $(E_n) \in \mathbf{Sp}^{\mathbb{P}^1}(\mathbf{PSh}_\bullet(\mathbf{Sm}/S))$ such that the maps $E_n \to \mathrm{R}\underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, E_{n+1})$ is an $\mathbb{A}^1$-local weak equivalence for all n. Here $\mathrm{R}\underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, -)$ is the derived functor of the right adjoint to $\mathbb{P}^1 \wedge -$ and the above map is the adjoint to the bonding map of E. A stable $\mathbb{A}^1$-equivalence is a map $E \to F$ of spectra such that for any Ω-spectrum V the induced map

$$\mathrm{Hom}_{\mathbf{SH}_p(S)}(F, V) \to \mathrm{Hom}_{\mathbf{SH}_p(S)}(E, V)$$

is a bijection.

A.3 The Deligne cohomology spectrum

Recall Theorem 2.3.4:

Theorem A.3.1. *(Burgos) There exists presheaves of complexes E_p on the category of smooth varieties over k, satisfying the following properties:*

1. *For every smooth variety V, we have $\mathrm{H}^n_{\mathrm{D}}(V, \mathbb{R}(p)) = \mathrm{H}^n(E_p(V))$ for all n and p.*

2. *The presheaves E_p are equipped with products $E_p \otimes E_{p'} \to E_{p+p'}$ which*

are (1) graded commutative on the level of complexes, and (2) associative up to homotopy. These products induce the usual product on the Deligne cohomology groups.

We will write S for the base scheme $\mathrm{Spec}\,(k)$, and it will be convenient to introduce the notation

$$D(p) = E_p[2p]$$

so that $\mathrm{H}^n_{\mathrm{D}}(V, \mathbb{R}(p)) = \mathrm{H}^{n-2p}(D(p)(V))$. The reason for shifting the complexes by $2p$ is that we will use them to construct a $\mathbb{P}^1$-spectrum rather than a $\mathbb{Q}(1)$-spectrum (recall that $\mathbb{P}^1$ in the $\mathbb{Q}$-linear setting corresponds to $\mathbb{Q}(1)[2]$), and the reason for moving the subscript p to a parenthesis is that we need to switch from cohomological to homological notation in order to apply the Dold-Kan correspondence. For this purpose, we write

$$\mathrm{D}_n(p) := \mathrm{D}^{-n}(p). \tag{A.2}$$

Now $\mathrm{D}_*(p)$ is a homological complex with differential of degree -1, and we have

$$\mathrm{H}_{n-2p}(\mathrm{D}_*(p)(V)) = \mathrm{H}^n_{\mathrm{D}}(V, \mathbb{R}(p)).$$

In order to have a complex of simplicial presheaves (as opposed to a complex of abelian groups), we use the Dold-Kan-equivalence

$$\mathcal{K} : \mathbf{Com}_{\geq 0}(\mathbf{Ab}) \rightleftarrows \Delta^{\mathrm{op}}(\mathbf{Ab}) : \mathcal{N}$$

between homological complexes concentrated in degrees ≥ 0 and simplicial abelian groups.

As usual, we write $\tau_{\geq n}$ for the good truncation of a complex.

Definition A.3.2. We write

$$\mathrm{D}_s(p) := \mathcal{K}(\tau_{\geq 0}\mathrm{D}_*(p)).$$

Lemma A.3.3. *For X smooth over S and any $k \geq 0$, $p \in \mathbb{Z}$ we have:*

$$\mathrm{Hom}_{\mathbf{Ho}_\bullet}(S^k \wedge X_+, \mathrm{D}_s(p)) = \mathrm{H}^{2p-k}_{\mathrm{D}}(X, p). \tag{A.3}$$

Proof: The statement does hold if we take the Hom-group in $\mathbf{Ho}_{sect,\bullet}$ instead of $\mathbf{Ho}_{\bullet}$:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Ho}_{sect,\bullet}}(S^k \wedge X_+, \mathcal{K}(\tau_{\geq 0}(\mathrm{D}_*(p)))) &= \pi_k \mathcal{K}(\tau_{\geq 0}(\mathrm{D}_*(p)(X))) \\ &= \mathrm{H}_k(\tau_{\geq 0}(\mathrm{D}_*(p)(X))) \\ &= \mathrm{H}_{\mathrm{D}}^{2p-k}(X,p).\end{aligned}$$

(We have used the identification $\pi_n(A,0) = \mathrm{H}_n(\mathcal{N}(A))$ for any simplicial abelian group and the fact that $\mathcal{K}(-)$, being a simplicial abelian group, is fibrant as a simplicial set.)

The presheaf $\mathrm{D}_s(p)$ is fibrant with respect to the $\mathbb{A}^1$- local model structure, since Deligne cohomology satisfies Nisnevich descent and is $\mathbb{A}^1$-invariant. Thus the Hom-groups agree when taken in $\mathbf{Ho}_{sect,\bullet}$ and $\mathbf{Ho}_{\bullet}$, respectively.

□

Via the Alexander-Whitney map (see [50]), the product on the $\mathrm{D}(p)$ complexes transfers to a product

$$\mathrm{D}_s(i) \wedge \mathrm{D}_s(j) \xrightarrow{\mu_{i,j}} \mathrm{D}_s(i+j)$$

and we will write $a \cdot_{\mathrm{D}} b$ for $\mu_{i,j}(a,b)$.

Before we can define the Deligne spectrum we need to record two additional facts about Deligne cohomology.

Lemma A.3.4. *For a section $x \in \mathrm{D}_0(X)$ satisfying $d_{\mathrm{D}}(x)(= dx) = 0$ and any two sections $y, z \in \mathrm{D}_*(X)$, we have*

$$x \cdot_{\mathrm{D}} (y \cdot_{\mathrm{D}} z) = (x \cdot_{\mathrm{D}} y) \cdot_{\mathrm{D}} z \tag{A.4}$$

and

$$x \cdot_{\mathrm{D}} y = y \cdot_{\mathrm{D}} x. \tag{A.5}$$

Proof: This is [10, Theorem 3.3]. (The commutativity statement is a special case of the graded commutativity which holds for all sections in the Burgos complexes, but the associativity does not hold in general without conditions on the sections.) $\square$

Lemma A.3.5. *Deligne cohomology satisfies the Künneth formula for $\mathbb{G}_m$, that is, for any smooth variety X, and any integers n, p, we have*

$$\mathrm{H}^n_{\mathrm{D}}(X \times \mathbb{G}_m, p) = \mathrm{H}^n_{\mathrm{D}}(X, p) \oplus \mathrm{H}^{n-1}_{\mathrm{D}}(X, p-1).$$

Proof: This follows from the long exact Mayer-Vietoris sequence associated to the standard cover of $X \times \mathbb{P}^1$ by $X \times \mathbb{A}^1$ and $X \times \mathbb{A}^1$ (intersecting in $X \times \mathbb{G}_m$), together with $\mathbb{A}^1$-homotopy invariance and the Künneth formula for $X \times \mathbb{P}^1$. $\square$

Definition A.3.6. The *Deligne cohomology spectrum* $\mathcal{E}_{\mathrm{D}}$ is the symmetric $\mathbb{P}^1$-spectrum consisting of the $\mathrm{D}_s(p)$ ($p \geq 0$), equipped with the trivial action of the symmetric group Σ_p, and bonding maps defined as the composition

$$\sigma_p : \mathbb{P}^1_S \wedge \mathrm{D}_s(p) \stackrel{c^* \wedge \mathrm{id}}{\to} \mathrm{D}_s(1) \wedge \mathrm{D}_s(p) \stackrel{\mu_{1,p}}{\to} \mathrm{D}_s(p+1).$$

Here c^* is the map induced by $c := c_1(\mathcal{O}_{\mathbb{P}^1}(1)) \in \mathrm{D}_0(1)(\mathbb{P}^1)$, the first Chern form of the bundle $\mathcal{O}(1)$ equipped with the Fubini-Study metric.

We equip $\mathcal{E}_{\mathrm{D}}$ with the following monoid structure: the product

$$\mu : \mathcal{E}_{\mathrm{D}} \wedge \mathcal{E}_{\mathrm{D}} \to \mathcal{E}_{\mathrm{D}}$$

is induced by the products $\mu_{p,p'} : \mathrm{D}_s(p) \wedge \mathrm{D}_s(p') \to \mathrm{D}_s(p+p')$. The unit map $\eta : \Sigma^{\infty}_{\mathbb{P}^1} S_+ \to \mathcal{E}_{\mathrm{D}}$ is defined in degree zero by the unit of the differential graded algebra $\mathrm{D}(0)$. In higher degrees, we put

$$\eta_p : (\mathbb{P}^1)^{\wedge p} \xrightarrow{(c^*)^{\wedge p}} \mathrm{D}_s(1)^{\wedge p} \xrightarrow{\mu} \mathrm{D}_s(p).$$

Equivalently, $\eta_p := \sigma_{p-1} \circ (\mathrm{id}_{\mathbb{P}^1} \wedge \eta_{p-1})$.

Proposition A.3.7. *The object $\mathcal{E}_{\mathrm{D}}$ is a symmetric $\mathbb{P}^1$-spectrum, with the structure of a weak ring spectrum.*

Proof: First, recall that c is a $(1,1)$-form which is invariant under Fr^*_∞ and under complex conjugation, so c is indeed an element of $\mathrm{D}_0(1)(\mathbb{P}^1)$.

Secondly, if we write ∞ for the immersion of the infinite point in $\mathbb{P}^1_S$, we have $\infty^* c = 0 \in \mathrm{D}_0(1)(S)$, since the pullback of c is a 2-form, but $\dim S = 0$. That is, c is a pointed map $(\mathbb{P}^1, \infty) \to (\mathrm{D}_0(1), 0)$. Thirdly, we have to show that the map

$$\begin{array}{rcl} \mathbb{P}^{1\wedge m} \wedge \mathrm{D}_s(n) & \stackrel{\mathrm{id}^{\wedge m-1} \wedge c^* \wedge \mathrm{id}}{\longrightarrow} & \mathbb{P}^{1\wedge m-1} \wedge \mathrm{D}_s(1) \wedge \mathrm{D}_s(n) \\ & \stackrel{\mu_{1,n}}{\to} & \mathbb{P}^{1\wedge m-1} \wedge \mathrm{D}_s(m+1) \\ & \to & \dots \\ & \to & \mathrm{D}_s(m+n) \end{array}$$

is a Σ_{m+n} equivariant map of presheaves on $\mathbf{Sm}/S$, i.e., invariant under permuting the m wedge factors $\mathbb{P}^1$. Given some map $f : U \to \mathbb{P}^{1\times m}$ with $U \in \mathbf{Sm}/S$, let $f_i : U \to \mathbb{P}^1$ be the i-th projection of f and $c_i := f_i^* c_1(\mathcal{O}_{\mathbb{P}^1}(1))$. Given some form $\omega \in \mathrm{D}(n)(U)$ (in some unspecified degree), the map is given by

$$(f, \omega) \mapsto c_1 \cdot_{\mathrm{D}} (c_2 \cdot_{\mathrm{D}} (\dots (c_m \cdot_{\mathrm{D}} \omega) \dots)).$$

Here $\cdot_{\mathrm{D}}$ denotes the product on $\mathrm{D}(*)$ (also denoted $\mu_{1,*}$). The forms $c_i \in \mathrm{D}_0(1)(U)$ are closed differential forms, so by Proposition A.3.4 the right hand expression is associative and commutative, i.e. invariant under the permutation action of Σ_m on $\mathbb{P}^{1\times m}$.

As in section 3.2 of the thesis, the monoid structure just defined gives $\mathcal{E}_{\mathrm{D}}$ the structure of a weak ring spectrum.

□

Lemma A.3.8. *The Deligne cohomology spectrum $\mathcal{E}_{\mathrm{D}}$ is an Ω-spectrum .*

Proof: We have to check that the adjoint map to the bonding map σ_p:

$$b_p : \mathrm{D}_s(p) \to \mathrm{R}\underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, \mathrm{D}_s(p+1)),$$

is a $\mathbb{A}^1$-local weak equivalence. As $\mathbb{P}^1$ is cofibrant and $\mathrm{D}_s(p+1)$ is fibrant, the non-derived $\underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, \mathrm{D}_s(p))$ is fibrant and agrees with $\mathrm{R}\underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, \mathrm{D}_s(p))$. The map is actually a sectionwise weak equivalence, i.e., an isomorphism in

$\mathbf{Ho}_{sect,\bullet}(S)$. To see this, it is enough to check that the map

$$\mathrm{D}_s(p)(U) \to \underline{\mathrm{Hom}_\bullet}(\mathbb{P}^1, \mathrm{D}_s(p+1)(U))$$

is a weak equivalence of simplicial sets for all $U \in \mathbf{Sm}/S$. The m-th homotopy group of the left hand side is $\mathrm{H}_{\mathrm{D}}^{2p-m}(U,p)$ (Lemma A.3.3), while π_m of the right hand simplicial set identifies with those elements of

$$\pi_m(\underline{\mathrm{Hom}}(\mathbb{P}^1 \times U, \mathrm{D}_s(p+1)) = \mathrm{H}_{\mathrm{D}}^{2(p+1)-m}(\mathbb{P}^1 \times U, p+1)$$

which restrict to zero when applying the restriction to the point $\infty \to \mathbb{P}^1$. By the Künneth formula for $X \times \mathbb{P}^1$, the two terms agree. □

Theorem A.3.9. *The spectrum $\mathcal{E}_{\mathrm{D}}$ represents Deligne cohomology in $\mathbf{SH}(S)$: for any smooth variety X over S, and any n, $m \in \mathbb{Z}$ we have*

$$\mathrm{Hom}_{\mathbf{SH}(S)}(S^n \wedge \Sigma^\infty_{\mathbb{P}^1}({\mathbb{P}^1_S}^{\wedge m} \wedge X_+), \mathcal{E}_{\mathrm{D}}) = \mathrm{H}_{\mathrm{D}}^{-n-2m}(X, -m).$$

(For $n < 0$ the left hand group is the same as $\mathrm{Hom}_{\mathbf{SH}(S)}(\Sigma^\infty_{\mathbb{P}^1} X_+, S^{-n} \wedge \mathcal{E}_{\mathrm{D}})$, taking into account that smashing with S^1 is invertible in $\mathbf{SH}(S)$, and likewise for $m < 0$.)

Proof: By Lemma A.3.8, $\mathcal{E}_{\mathrm{D}}$ is an Ω-spectrum. Hence the claim follows from Lemma A.3.3, completely analogous to the proof of Theorem 3.2.4. □

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