Motivic representation rings

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Introduction

Motives are thought of as a “universal cohomology theory” for schemes.

The idea is that any reasonable cohomology functor from schemes to some linear category $\mathcal{V}$ factors via the category of motives:

$$\text{Schemes} \longrightarrow \text{Motives} \longrightarrow \mathcal{V}$$

Can make this precise in many ways, but categories of motives are difficult to understand!

Idea: Decategorify!

Introduction

Example: Consider a finite group G, and the category $Rep_{\mathbb{C}}(G)$.

Can take the *Grothendieck ring* $K_0(Rep_{\mathbb{C}}(G))$.

It is a commutative ring, generated by the irreducible representations.

Addition $\leftrightarrow$ direct sum of representations

Multiplication $\leftrightarrow$ tensor product of representations.

Exterior powers give extra algebraic structure, giving us a *lambda-ring*, with lambda-operations, Adams operations, and more.

Introduction

Representation rings are sometimes described only as commutative rings.

Example: The Grothendieck ring of $Rep_{\mathbb{C}}(S_3)$ is isomorphic to $\mathbb{Z}[X, Y]/(XY - Y, X^2 - 1, Y^2 - X - Y - 1)$

(Here 1 is the trivial rep, X is the sign rep (dim 1), and Y is the 2-dimensional irrep.)

But this is bad - the lambda-ring structure is important!

Introduction

Example: Consider (complex) representations of a compact connected complex Lie group.

The Grothendieck ring is generated *as a ring* by elements in one-to-one correspondence with the nodes of the associated Dynkin diagram.

The Grothendieck ring is generated *as a lambda-ring* by elements in one-to-one correspondence with the *arms* of the Dynkin diagram.

There are also structure theorems characterizing which lambda-rings can occur as representation rings of these Lie groups, as well as a theorem saying that the Lie group itself is determined by the Grothendieck ring together with one extra piece of data.

Introduction

General categorical framework for categories like $Rep_{\mathbb{C}}(G)$: *Tannakian categories*.

For any such category $\mathcal{T}$, the Grothendieck ring $K_0(\mathcal{T})$ is a lambda-ring.

Main question: Can we give explicit descriptions of these lambda-rings in a way that clearly captures also the lambda-ring structure?

Didactical problem

Question: How teach algebraic structures (to high-school students)?

Finite groups: Use permutation representations

Commutative rings: Use polynomials

Associative algebras: Use matrices

Lie algebras: Use matrices

Lambda-rings: Use ???

Didactical problem

Question: How teach algebraic structures (to high-school students)?

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Lambda-rings: Use **Tannakian symbols**

Lambda-rings

Let R be a torsion-free commutative ring. A lambda-structure on R is an infinite sequence of ring homomorphisms ψ^1, ψ^2, ... from R to R satisfying the following axioms:

1. $\psi^1(x) = x$ for all $x \in R$.
2. $\psi^m(\psi^n(x)) = \psi^{mn}(x)$ for all m, n and all $x \in R$.
3. $\psi^p(x) \equiv x^p \pmod{pR}$ for all prime numbers p and all $x \in R$.

Tannakian symbols

Let M be a commutative monoid (set with a binary operation that is associative, commutative, and has an identity element).

Example: $M = \mathbb{C}^*$ (under multiplication).

A *finite multiset* is a finite unordered list of elements (repeated elements allowed).

A *Tannakian symbol* (with values in M) is an ordered pair of finite multisets with elements taken from M. We require the multisets to be disjoint.

Notation: $\frac{A}{B}$

Example: $\frac{\{2,2,5,5\}}{\{1,1,1\}}$

Tannakian symbols

Operations on Tannakian symbols (examples):

$$\frac{\{5\}}{\{1,-1\}} \oplus \frac{\{1,1,1\}}{\{-1\}} = \frac{\{5,\cancel{1},1,1\}}{\{\cancel{1},-1,-1\}} = \frac{\{5,1,1\}}{\{-1,-1\}}$$

$$\frac{\{5\}}{\{1,-1\}} \otimes \frac{\{10\}}{\{3,7\}} = \frac{\{50,3,7,-3,-7\}}{\{15,35,10,-10\}}$$

$$\psi^2\left(\frac{\{-1,-1,2,5\}}{\{1,-2,7\}}\right) = \frac{\{\cancel{(-1)^2},(-1)^2,\cancel{2^2},5^2\}}{\{\cancel{1^2},\cancel{(-2)^2},7^2\}} = \frac{\{1,25\}}{\{49\}}$$

Tannakian symbols

We write $TS(M)$ for the set of Tannakian symbols with values in M.

Theorem: $TS(M)$, with the above operations, is a lambda-ring.

As a commutative ring, it is isomorphic to the monoid algebra of M.

TS is a functor from commutative monoids to lambda-rings.

Let U be a set. We write $TS(M)^U$ for the set of functions from U to $TS(M)$ (think of this as vectors of symbols, indexed by U).

Tannakian symbols

Main conjecture:

Let $\mathcal{T}$ be a Tannakian category, with Grothendieck ring $K_0(\mathcal{T})$. Let L be any sub-lambda-ring or quotient lambda-ring of $K_0(\mathcal{T})$.

- There exists a monoid M, a set U, and an *injective lambda-ring homomorphism*

$$L \hookrightarrow TS(M)^U$$

.

- If L is finitely generated, then U may be taken to be finite.
- There exists a practical algorithm associated to L that takes an element of $TS(M)^U$ as input and determines whether it comes from L.

Tannakian symbols

The most interesting aspect here is not to prove the conjecture, but to work out concrete examples.

Many relations to classical problems of number theory, algebraic geometry and representation theory.

Motives

Want to talk about motives over $Spec(\mathbb{Z})$ (this is the most interesting and most complicated case of motives, but there are many others).

Let X be a scheme of finite type over $Spec(\mathbb{Z})$. In practice, this means we consider a set polynomial equations with integer coefficients.

To the scheme X we can associate a *motive* $h(X)$, and more refined "weight pieces" $h^i(X)(m)$. Here i and m are two integers.

Motives

Each motive $h^i(X)(m)$ splits into a direct sum of irreducibles. These summands (as we vary X, i and m) generate a lambda-ring, which is the Grothendieck ring of motives (over $Spec(\mathbb{Z})$).

We can construct a map from this lambda-ring to $TS(M)^U$, where $M = \mathbb{C}^*$, and U is the set of prime numbers.

The data we use is the same data found in the Hasse-Weil zeta function of X, but organised into "combinatorial" objects rather than a complex-analytic function.

Examples

Example: The scheme X defined by the equation $y^2 + y = x^3 - x^2$ (an elliptic curve).

At the prime $p = 2$, the symbol becomes:

$$\frac{\{-1+i, -1-i\}}{\{1, 2\}}$$

The absolute values of these numbers are $\sqrt{2}$ and $\sqrt{2}$ upstairs, and 1 and 2 respectively downstairs. From this we know the dimension and the Betti numbers of the scheme.

Examples

Let X_κ be the "quartic Dwork family", i.e. the (projective) scheme defined by the equation

$$x^4 + y^4 + z^4 + w^4 = 4\kappa xyzw$$

where κ is a integer-valued parameter.

The scheme X_κ comes with a natural action of the group $\mathbb{Z}/4 \times \mathbb{Z}/4$. Taking the quotient scheme by this group action and resolving singularities yields a new scheme Y_κ, called the mirror of X_κ.

Computations borrowed from a presentation by Ursula Whitcher, using code by Edgar Costa.

Examples

Look at $p = 41$. For $\kappa = 2$, the scheme X_κ has symbol:

$$\{1, 41, 41, 41, 41, -41,$$

$$-41, \ldots, -41, \frac{25 - 8\sqrt{66}i}{2}, \frac{25 + 8\sqrt{66}i}{2}, 1681\}/\emptyset$$

Here there are 4 copies of the number 41 and 16 copies of the number -41.

For the mirror variety Y_2, we get

$$\{1, 41, 41, \ldots, 41, -41, \frac{25 - 8\sqrt{66}i}{2}, \frac{25 + 8\sqrt{66}i}{2}, 1681\}/\emptyset$$

with 19 copies of the number 41, a single copy of the number -41.

Examples

Still working with $p = 41$, for the case $\kappa = 3$ we get for X_3:

$$\{1, 41, 41, \ldots, 41, -39 + 4\sqrt{10}i, -39 - 4\sqrt{10}i, 1681\}/\emptyset$$

with 20 copies of the number 41.

And this time, the Tannakian symbol for the mirror variety Y_3 is

$$\{1, 41, 41, \ldots, 41, -39 + 4\sqrt{10}i, -39 - 4\sqrt{10}i, 1681\}/\emptyset$$

with 20 copies of 41.

Completely identical symbols!

Examples

Question: Is there an explicit algebraic operator on $K_0(Mot)$ that sends the class of a variety to the class of its mirror?

There are literally hundreds of questions like this about explicit constructions in Grothendieck rings, which we can investigate using Tannakian symbols.

Can use these symbols in any situation in which we have rational zeta functions, or zeta functions with Euler products, or a Tannakian category, or a category with a symmetric monoidal functor to a Tannakian category.

Wonderful didactical tool!

Examples

One final example: For any element of $K_0(Mot)$, and any integer n, there should be a "special value formula".

The simplest case is Euler's Basel problem. Take the scheme $x = 0$ and the integer $n = 2$. We get

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

Examples

Two final remarks on work in progress:

1. New lambda-ring structures on multiplicative functions in elementary number theory. Clarifies a multitude of strange identities, some going back to Ramanujan.

2. First major application of infinity-categories to number theory: Proof of the Tamagawa number conjecture by Gaitsgory and Lurie.

One ingredient: Zeta functions of stacks (not rational!). For these we can make computations using *generalised* Tannakian symbols.

Thank you!