Cohomology theories in motivic stable homotopy theory

Andreas Holmstrom

Universitetet i Oslo

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Outline

Algebraic topology

Question:
What is a cohomology theory?

Answer:
A cohomology theory is a functor from spaces into graded abelian groups, which is representable by an object in the stable homotopy category **SH**.

$$Spaces \xrightarrow{\Sigma^\infty} \mathbf{SH} \xrightarrow{Hom(-,\Sigma^n E)} Ab^*$$

Equivalently, it is a sequence of functors satisfying the (generalized) Eilenberg-Steenrod axioms.

Algebraic topology

Examples:
Singular cohomology, various versions of K-theory, complex cobordism, BP-theory, Morava K-theories and E-theories, tmf, ...

Topologists have studied the structure of **SH** for a long time, and have developed many tools for working with cohomology theories.

Properties of a cohomology theory are a reflection of properties of the representing spectrum. Basic examples:

- Multiplicative cohomology theories correspond to ring spectra
- Oriented cohomology theories correspond to MU-algebras

Algebraic geometry

Question:
What is a cohomology theory?

Answer:
?????

- There is no framework in which all cohomology theories fit.
- There are no textbooks or surveys titled "Cohomology in algebraic geometry" - the subject is to big!
- Problem, not for the experts in algebraic geometry, but for students.
- Lack of framework makes it harder to make progress on cohomology in settings more general than algebraic varieties.

Algebraic geometry

Many examples of cohomology

1-motives
Absolute cohomology of stacks
Additive Chow groups
Additive K-theory
Algebraic cycle groups
Algebraic elliptic cohomology
Algebraic K-homology
Algebraic K-theory with compact supports
Algebraic Morava K-theory
Analytic cyclic cohomology
Arakelov motivic cohomology
Arithmetic Chow groups
Arithmetic homology
Artin motives
Bi-relative algebraic K-theory
Bi-relative topological cyclic homology
Birelative homology

Absolute cohomology
Absolute Hodge cohomology
Additive higher Chow groups
Algebraic cobordism
Algebraic de Rham cohomology
Algebraic G-theory
Algebraic K-theory
Algebraic L-theory
Amitsur cohomology
Arakelov Chow groups
Archimedean cohomology
Arithmetic cohomology
Arithmetic K-theory
Betti cohomology
Bi-relative cyclic homology
Big de Rham-Witt cohomology
Bivariant cycle cohomology

Algebraic geometry

Many examples of cohomology

- Bivariant cycle homology
- Bloch-Ogus cohomology
- Borel-Moore motivic homology
- Bounded K-theory
- Bredon style cohomology of stacks
- Cech cohomology
- Chow groups of stacks
- Chow motives
- Cisinski-DŐglise motives
- CM motives
- Cohomology with compact supports
- Completed cohomology
- Continuous etale cohomology
- Cristalline Deligne cohomology
- Cyclic homology
- de Rham-Witt cohomology
- Deligne homology
- Bivariant K-theory
- Borel-Moore homology
- Boundary cohomology
- Bousfield's united K-theory
- cdh-cohomology
- Chow groups
- Chow groups with coefficients
- Chow-Witt groups
- Classical motives
- Cohomology of coherent sheaves
- Compact K-theory
- Continuous cohomology
- Continuous hypercohomology
- Crystalline cohomology
- de Rham cohomology
- Deligne cohomology
- Deligne-Beilinson cohomology

Algebraic geometry

Many examples of cohomology

- Deninger cohomology
- Dwork cohomology
- Eichler cohomology
- Elliptic Bloch groups
- Equivariant Cech cohomology
- Equivariant higher Chow groups
- Equivariant pretheory
- Etale BP2
- Etale cohomology
- Etale cohomology with compact support
- Etale K-theory
- Etale Morava K-theory
- f-cohomology
- Finite polynomial cohomology
- Flat cohomology
- Fontaine-Messing cohomology
- Derivator K-theory
- Dynamical cohomology
- Eisenstein cohomology
- Equivariant algebraic K-theory
- Equivariant etale cohomology
- Equivariant operational Chow groups
- Equivariant smooth Deligne cohomology
- Etale cobordism
- Etale cohomology of rigid analytic spaces
- Etale homology
- Etale K-theory of ring spectra
- Etale motivic cohomology
- Faltings cohomology
- Finite-dimensional motives
- Flat homology
- Formal cohomology

Algebraic geometry

Many examples of cohomology

- Friedlander-Suslin cohomology
- G-theory
- Generalized de Rham cohomology
- Generalized sheaf cohomology
- Geometric cohomology
- Gorenstein cohomology
- Grothendieck-Witt groups
- Hermitian-holomorphic Deligne cohomology
- Higher arithmetic Chow groups
- Higher Chow groups
- Hodge-Witt cohomology
- Homology of schemes
- Hyodo-Kato cohomology
- Infinitesimal cohomology
- Intersection Chow groups
- K'-theory
- Frobenius cohomology
- G-theory of algebraic stacks
- Generalized etale cohomology
- Generic cohomology
- Gillet cohomology
- Grothendieck motives
- Hermitian K-theory
- Higher Arakelov Chow groups
- Higher arithmetic K-theory
- Hochschild homology
- Holomorphic K-theory
- Homotopy K-theory
- Hypercohomology
- Infinitesimal K-theory
- Intersection cohomology
- K-theory of stacks

Algebraic geometry

Many examples of cohomology

- K-theory with coefficients
- Karoubi L-theory
- Kato homology
- l-adic cohomology
- Laumon 1-motives
- Local cohomology
- Log Betti cohomology
- Log crystalline cohomology
- Log etale cohomology
- Logarithmic cohomology
- Milnor K-theory
- Mixed Tate motives
- Modified syntomic cohomology
- Morphic cohomology
- Motives for absolute Hodge cycles
- Motivic cohomology
- K-theory with supports
- Karoubi-Villamayor K-theory
- l-adic algebraic K-theory
- l-adic parabolic cohomology
- Lawson homology
- Locally analytic cohomology
- Log convergent cohomology
- Log de Rham cohomology
- Log Hodge groups
- Logarithmic Hodge-Witt cohomology
- Mixed motives
- Mixed Weil cohomology
- Monsky-Washnitzer cohomology
- Motives
- Motivic cobordism
- Motivic cohomology of stacks

Algebraic geometry

Many examples of cohomology

Motivic cohomology with compact supports
Negative cyclic homology
Nisnevich cohomology
Non-standard Őtale cohomology
Noncommutative motives
Orientable cohomology
Oriented Chow groups
Oriented homology
p-adic cobordism
p-adic etale cohomology
Parabolic cohomology
Positive cohomology
Primitive cohomology
Quillen K-theory
Relative Bloch groups
Relative rigid cohomology
Rigid syntomic cohomology

Motivic homology
Negative K-theory
Nisnevich K-theory
Nonabelian cohomology
Operational Chow groups
Oriented Borel-Moore homology
Oriented cohomology
p-adic Chow motives
p-adic cohomology
p-adic K-theory
Periodic cyclic homology
Pretheory
Pure motives
Relative algebraic K-theory
Relative K-theory
Rigid cohomology
Rost's cycle modules

Algebraic geometry

Many examples of cohomology

- Schmidt homology
- Semi-topological K-theory
- Sharp motives
- Smooth cohomology
- Stable cohomology
- Suslin homology
- t-motives
- Topological cycle cohomology
- Topological Hochschild homology
- Twisted duality theory
- Voevodsky motives
- Waldhausen K-theory
- Weil cohomology
- Weil-etale motivic cohomology
- Witt cohomology with supports
- Witt vector cohomology
- Semi-topological K-homology
- Sharp cohomology
- Sheaf cohomology
- Smooth Deligne cohomology
- Stack cohomology
- Syntomic cohomology
- Thomason-Trobaugh K-theory
- Topological cyclic homology
- Twisted cohomology
- Unramified cohomology
- Volodin K-theory
- Weak Hodge cohomology
- Weil-etale cohomology
- Witt cohomology
- Witt groups
- Zariski cohomology

Algebraic geometry

Properties of cohomology theories

Pick your favourite cohomology theory. It probably satisfies a long list of formal properties which makes it useful for applications.

Contravariant functoriality	Chern classes
Covariant functoriality	Cycle classes
Poincaré duality	Product structure
Mayer-Vietoris	Descent for various topologies
Projective bundle formula	Blow-up formula
Kunneth formula	Finiteness properties
Vanishing properties	Versions of rigidity
Versions of purity	Localization sequences
Base change theorems	+ + + + +

Algebraic geometry

Questions:

- How can we understand which formal properties to expect from a randomly picked theory?
- Can we characterize the theories satisfying such-and-such properties?
- For example, which of the above theories satisfy Poincaré duality?
- Which of the theories have pushforwards? (i.e. covariant functoriality for proper morphisms)

Algebraic geometry

Axiom systems

- Weil cohomology theory
- Bloch-Ogus theory
- Oriented theory
- Geometric theory
- Pretheory
- ...

Question:
How can we understand these axiom systems in a unified way?

Algebraic geometry

Question:
What is a cohomology theory?

Answer:
?????

Many possible answers, none of them completely satisfactory.

Today: Will present the best answer given so far, namely motivic stable homotopy theory
(Morel, Voevodsky, Ayoub, Cisinski, Déglise, . . .)

The category $\mathbf{SH}(S)$

For any finite-dimensional scheme S, there is an associated stable homotopy category $\mathbf{SH}(S)$.

Sketch of construction:

- Take the category of smooth S-schemes, with the Nisnevich topology
- Let
$$\mathrm{Spaces}(S) = \Delta^{op}\mathrm{Sh}_{\bullet}(\mathrm{Sm}/S)$$
be the category of pointed simplicial sheaves on this category
- Let $T \in \mathrm{Spaces}(S)$ be a "motivic sphere" (e.g. $\mathbb{A}^1/\mathbb{G}_m$)
- Let $\mathrm{Spectra}(S)$ be the category of symmetric T-spectra
- $\mathbf{SH}(S)$ is the homotopy category of $\mathrm{Spectra}(S)$

Six functors formalism

Key difference between algebraic geometry and topology!

Six functors formalism

- For any scheme X, the triangulated category $\mathbf{SH}(X)$ is closed symmetric monoidal (has tensor product and internal Hom).
- For any morphism $f : Y \to X$, there is a pair of adjoint functors

$$f^* : \mathbf{SH}(X) \rightleftarrows \mathbf{SH}(Y) : f_*$$

  and f^* is a monoidal functor.
- For any separated morphism of finite type $f : Y \to X$, there is a pair of adjoint functors

$$f_! : \mathbf{SH}(Y) \rightleftarrows \mathbf{SH}(X) : f^!$$

- In addition to the shift functor $[n]$, there is another invertible operator (m), called "twist". (Here $m, n \in \mathbb{Z}$)
- These "six functors" satisfy a long list of properties.

Cohomology theory represented by a spectrum

For E in $\mathbf{SH}(S)$, and X a finite type S-scheme, we define E-cohomology of X by:

- $E^n(X, p) = \mathrm{Hom}_{\mathbf{SH}(S)}(\mathbf{1}, f_* f^* E(p)[n])$

We can also define other theories:

- Homology: $E_n(X, p) = \mathrm{Hom}(\mathbf{1}, f_! f^! E(-p)[-n])$
- Cohomology with compact support:

$$E_c^n(X, p) = \mathrm{Hom}(\mathbf{1}, f_! f^* E(p)[n])$$

- Borel-Moore homology:

$$E_n^{BM}(X, p) = \mathrm{Hom}(\mathbf{1}, f_* f^! E(-p)[-n])$$

Algebraic geometry

Main points of this talk:

1. Cohomology theories for S-schemes should be representable by objects in $\mathbf{SH}(S)$.
2. Formal properties of a cohomology theory should be governed by
   - The six functors formalism
   - The "geography" of $\mathbf{SH}$

Questions:

- **To what extent is this true?**
- **To what extent is it helpful?**

Representability theorems

General "theorems":

- Any cohomology theory which is $\mathbb{A}^1$-invariant and satisfies Nisnevich descent should be representable (Morel?)
- Nisnevich descent should be automatically satisfied for any theory defined as sheaf cohomology
- Any mixed Weil cohomology theory is representable (Cisinski and Déglise)
- Any Bloch-Ogus theory should be representable (Levine?)
- Any oriented theory should be representable

Making all of these precise is not completely trivial.

Example: Representability of Bloch-Ogus theories

Classical axioms:

We note that, if $W \subset W' \subset X$ are closed subsets of $X \in \mathbf{Sm}_k$, we have the natural map

$$H^n_{\Gamma,W}(X,m) \to H^n_{\Gamma,W'}(X,m)\,.$$

Definition 30 We say that $\Gamma(*)$ defines a *Bloch–Ogus cohomology theory* if

1. The product μ is associative and commutative in $D(\mathrm{Sh}^{Zar}(k))$.
2. $\Gamma(*)$ is *homotopy invariant*: $p^* : H^*_\Gamma(X,m) \to H^*_\Gamma(X \times \mathbb{A}^1, m)$ is an isomorphism for all m.
3. $\Gamma(*)$ satisfies *purity*: Let $W \subset X$ be a closed subset, with $X \in \mathbf{Sm}_k$. If $\operatorname{codim}_X W \geq q$ for some integer q, then $H^p_{\Gamma,W}(X,q) = 0$ for $p < 2q$.
4. $\Gamma(*)$ admits natural *cycle classes*: Let $W \subset X$ be an irreducible closed codimension q subset with X in $\mathbf{Sm}_k$. Then there is a *fundamental class* $[W] \in H^{2q}_{\Gamma,W}(X,q)$ satisfying:
   a) *Naturality*: Let $z := \sum_i n_i W_i$ be in $z^q(X)$, let W be the support of z, and set $\mathrm{cl}(z) = \sum_i n_i [W_i] \in H^{2q}_{\Gamma,W}(X,q)$. Let $f : Y \to X$ be a morphism in $\mathbf{Sm}_k$ such that $f^{-1}(W)$ has codimension q on Y. Then
   $$f^*(\mathrm{cl}(z)) = \mathrm{cl}(f^*(z)) \in H^{2q}_{\Gamma,f^{-1}(W)}(Y,q)\,.$$
   b) *Gysin isomorphism*: Suppose that $W \subset X$ is a pure codimension q closed subset, with X and W in $\mathbf{Sm}_k$. Suppose that the inclusion $i : W \to X$ is split by a smooth morphism $p : X \to W$. Then the composition
   $$H^n_\Gamma(W,m) \xrightarrow{p^*} H^n_\Gamma(X,m) \xrightarrow{\cup [W]} H^{n+2q}_{\Gamma,W}(X,m+q)$$
   is an isomorphism.
   c) *Products*: For $X_i \in \mathbf{Sm}_k$, $z_i \in z^{q_i}(X_i)$ with support W_i, $i = 1,2$, we have
   $$\mathrm{cl}(z_1 \times z_2) = p_1^* \,\mathrm{cl}(z_1) \cup p_2^* \,\mathrm{cl}(z_2)$$
   in $H^{2q_1+2q_2}_{\Gamma,W_1\times W_2}(X_1 \times_k X_2, q_1+q_2)$.
5. *Coefficients*: For $p : X \to \operatorname{Spec} k$ in $\mathbf{Sm}_k$, X irreducible, the map
   $$p^* : H^0_\Gamma(\operatorname{Spec} k, 0) \to H^0_\Gamma(X,0)$$
   is an isomorphism.

The functor $X \mapsto \oplus_{p,q} H^p_\Gamma(X,q)$ is called the *Bloch–Ogus theory on* Sm_k *represented by* $\Gamma(*)$. The ring $H^0_\Gamma(\operatorname{Spec} k, 0)$ is called the *coefficient ring* of the theory Γ.

Example: Representability of Bloch-Ogus theories

New tentative definition:

A Bloch-Ogus theory is an $\mathrm{H}\mathbb{Z}$*-algebra.*

Much simpler!!

Similar simplifications should be possible for other axioms systems.

More simplified axiom systems

- An oriented theory is an MGL-algebra. (Vezzosi?)
- A Beilinson theory is an $\mathrm{H}\mathbb{Q}$-module
- (Tentative) A Weil theory is an $\mathrm{H}\mathbb{Q}$-algebra E such that $E(1) \simeq E$.
- (Tentative) An ordinary theory is a theory whose cohomology groups are functorial with respect to correspondences. (For an oriented theory, this should be equivalent to the formal group law being the additive one.)

More representability theorems

Specific examples:

- Algebraic K-theory (KGL)
- Voevodsky's motivic cobordism (MGL)
- Motivic cohomology ($\mathrm{H}\mathbb{Z}$)
- ℓ-adic cohomology, algebraic de Rham cohomology, rigid cohomology (Cisinski-Déglise)
- Hermitian K-theory (Hornbostel)
- Deligne cohomology (Holmstrom-Scholbach)

The geography of **SH**

With the above tentative definitions we can attempt to draw a "map of all cohomology theories in algebraic geometry".

Many formal properties of cohomology theories should be governed by where they are located on such a map.

A map of cohomology theories (CTs) for schemes

All CTs

Non-$\mathbf{A}^1$-invariant CTs
E.g. arithmetic Chow gps, crystalline coh., Weil-etale coh., p-adic etale coh., +++++

CTs not satisfying Nisnevich descent
E.g. truncated Deligne coh., various presheaf coh. gps.

CTs related to noncommutative motives
Many versions of cyclic and Hochschild homology, algebraic K-theory, bivariant CTs, +++++

Representable CTs (arrows are inclusions)

Representable **Q**-CTs -----> Representable ordinary CTs

Λ All **Q**-sheaf coh? Λ All sheaf coh?

Mixed Weil CTs ---> Beilinson CTs ------> Bloch-Ogus CTs -----------> Orientable CTs

Mixed Weil CTs	Beilinson CTs	Bloch-Ogus CTs	Orientable CTs
Etale **Z**/*l*-coh.	Rational motivic	Motivic coh.	Algebraic K-theory
Alg. de Rham	Deligne R-coh.	Morphic coh.	Motivic cobordism
Rigid coh.	Arakelov motivic	Deligne **Z**-coh.	+++
Betti coh.	+++	+++	

Omitted: Chromatic theory
Motivic Morava K-th
Motivic BP, +++

Omitted are also CTs for stacks, for log schemes, and other more general algebraic geometry objects

Properties of CTs

Properties we would like to understand:

Contravariant functoriality	Chern classes
Covariant functoriality	Cycle classes
Poincaré duality	Product structure
Mayer-Vietoris	Descent for various topologies
Projective bundle formula	Blow-up formula
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Versions of purity	Localization sequences
Base change theorems	+ + + + +

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Properties of CTs

Example 1: Pushforwards

Proposition: Let E be a Beilinson cohomology theory. Let $f : X \to Y$ be a smooth projective morphism of relative dimension d. Then there are functorial pushforward maps

$$f_* : E^n(X, p) \to E^{n-2d}(Y, p-d).$$

Six functors formalism: further properties

- For any morphism f separated of finite type, there is a natural transformation $f_! \to f_*$ which is an isomorphism if f is proper.
- If f is an open immersion, we have $f^! = f^*$.
- For an $\mathrm{H}\mathbb{Q}$-module E, we have more generally that $f^* = f^!(-d)[-2d]$

Properties of CTs

Example 1: Pushforwards Proposition: Let E be a Beilinson cohomology theory. Let $f : X \to Y$ be a smooth projective morphism of relative dimension d. Then there are functorial pushforward maps

$$f_* : E^n(X, p) \to E^{n-2d}(Y, p-d).$$

Proof: Let p_X and p_Y be the structural morphisms of X and Y respectively. It is enough to construct a map $p_{X*}p_X^* = p_{Y*}f_*f^*p_Y^* \to p_{Y*}p_Y^*(d)[2d]$. But $f_* = f_!$ since f is proper, and we have $f^* = f^!(-d)[-2d]$ for any Beilinson module whenever f is smooth and quasi-projective. So for f smooth and projective, the counit $f_!f^! \to id$ gives the desired map.

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Properties of CTs

Example 2: Localization sequences

For a closed immersion $Z \to X$ with complementary open immersion $j : U \to X$, we would like to relate the cohomology groups of Z, X and U in some way.

Six functors formalism: Localization

Let $Z \to X$ be a closed immersion with complementary open immersion $j : U \to X$. Then the following holds:

- The functor $j_!$ is left adjoint to j^*.
- The functor i_* is left adjoint to $i^!$.
- The functors $j_!$ and i_* are fully faithful.
- We have $i^* j_! = 0$
- For any object M of $\mathbf{SH}$, there are natural distinguished triangles

$$j_! j^! M \to M \to i_* i^* M \to j_! j^! M[1]$$

and

$$i_! i^! M \to M \to j_* j^* M \to i_! i^! M[1]$$

where the maps are given by units and counits of the relevant adjoint pairs of functors. These triangles are referred to as localization triangles.

A map of cohomology theories (CTs) for schemes

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Properties of CTs

Example 3: Descent properties

- Any representable theory satisfies Nisnevich descent, so in particular has a long exact Mayer-Vietoris sequence:

$$\dots E^n(X,p) \to E^n(U,p) \oplus E^n(U,p) \to E^n(U \cap V,p) \to E^{n+1}(X,p) \to$$

- Any Beilinson theory satisfies h-descent, so in particular Galois descent and faithfully flat descent.

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Properties of CTs

Example 4: Cycle classes

For smooth varieties over a field, there is a comparison isomorphism between Chow groups $CH^n(X)$ and the motivic cohomology groups $\mathrm{H}^{2n}_M(X, n)$.
Therefore any HZ-algebra (Bloch-Ogus theory) admits cycle classes.

A map of cohomology theories (CTs) for schemes

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What about the non-representable CTs?

- Often there are comparison theorems between representable cohomology theories and non-representable theories. These can be used to understand non-representable theories via results in motivic stable homotopy theory.
- Crystalline cohomology (non-representable) agrees with rigid cohomology (representable) for smooth *projective* varieties. Hence crystalline cohomology is a Weil cohomology for these varieties.
- Truncated Deligne cohomology groups $\mathrm{H}^n_D(X, p)$ (non-representable) agree with ordinary Deligne cohomology for $n \leq 2p$. Hence truncated Deligne cohomology admits cycle maps.

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E.g. arithmetic Chow gps, crystalline coh., Weil-etale coh., p-adic etale coh., +++++

CTs not satisfying Nisnevich descent
E.g. truncated Deligne coh., various presheaf coh. gps.

CTs related to noncommutative motives
Many versions of cyclic and Hochschild homology, algebraic K-theory, bivariant CTs, +++++

Representable CTs (arrows are inclusions)

Omitted: Chromatic theory
Motivic Morava K-th
Motivic BP, +++

Representable **Q**-CTs -----> Representable ordinary CTs

Λ All **Q**-sheaf coh? | Λ All sheaf coh?

Mixed Weil CTs ---> Beilinson CTs ------> Bloch-Ogus CTs -----------> Orientable CTs

Mixed Weil CTs	Beilinson CTs	Bloch-Ogus CTs	Orientable CTs
Etale **Z**/*l*-coh.	Rational motivic	Motivic coh.	Algebraic K-theory
Alg. de Rham	Deligne R-coh.	Morphic coh.	Motivic cobordism
Rigid coh.	Arakelov motivic	Deligne **Z**-coh.	+++
Betti coh.	+++	+++	

Omitted are also CTs for stacks, for log schemes, and other more general algebraic geometry objects

Final remarks

- An *arithmetic scheme* is a scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$. Cohomology for such schemes is very poorly understood.
- Thanks to Ayoub, Déglise and Cisinski, motivic stable homotopy theory can be useful for studying cohomology of such schemes, but it is not good enough for all purposes.
- **Question:** What is the "right" notion of stable homotopy theory for arithmetic schemes???