Motivic Symbols and Classical Multiplicative Functions

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$$\frac{\{1, p^k\}}{\varnothing}$$

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Abstract

A fundamental tool in number theory is the notion of an arithmetical function, i.e. a function which takes a natural number as input and gives a complex number as output. Among these functions, the class of multiplicative functions is particularly important. A multiplicative function has the property that all its values are determined by the values at powers of prime numbers.

Inside the class of multiplicative functions, we have the smaller class of completely multiplicative functions, for which all values are determined simply by the values at prime numbers.

Classical examples of multiplicative functions include the constant function with value 1, the Liouville function, the Euler totient function, the Jordan totient functions, the divisor functions, the Möbius function, and the Ramanujan tau function. In higher number theory, more complicated examples are given by Dirichlet characters, coefficients of modular forms, and more generally by the Fourier coefficients of any motivic or automorphic L-function.

It is well-known that the class of all arithmetical functions forms a commutative ring if we define addition to be the usual (pointwise) addition of functions, and multiplication to be a certain operation called Dirichlet convolution. A commutative ring is a kind of algebraic structure which, just like the integers, has both an addition and a multiplication operation, and these satisfy various axioms such as distributivity: $f \cdot (g + h) = f \cdot g + f \cdot h$ for all f, g and h.

Restricting attention to the class of multiplicative functions, it is natural to ask whether it also can be equipped with some natural algebraic structure. We have not found any number theory textbook in which any such structure is mentioned, beyond the abelian group structure given by Dirichlet convolution. However, there are abstract theoretical reasons (related to motives) for believing that the class of multiplicative functions should form a so-called lambda-ring, with Dirichlet convolution now taking the role of addition. A lambda-ring is a kind of commutative ring with infinitely many extra operations, and such rings appear naturally in several different mathematical fields, including representation theory and K-theory.

The aim of this paper is to prove that a certain class of multiplicative functions (which we call classical) really do form a lambda-ring. This is our main theorem.

In order to prove this theorem, we construct a functor from the category of commutative monoids to the category of lambda-rings. The main ingredient in this construction is a simple set-theoretic notion which we call a motivic symbol. The functor itself is denoted by MS. Under certain hypotheses we also show how to associate a linearly recursive sequence to a motivic symbol, called its corresponding sequence.

Then we study many different multiplicative functions known from textbooks in number theory, and show how to assign motivic symbols to such functions, again under certain hypotheses.

Finally, we give a precise definition of what we mean by a classical multiplicative function, and we prove that there is a bijective correspondence between such functions and the lambda-ring of motivic symbols obtained from a monoid which is isomorphic to a direct product of the additive monoid of natural numbers with a cyclic group of order 2.

Under this correspondence, Dirichlet convolution of functions corresponds to sum of motivic symbols. Also, pointwise multiplication of two functions corresponds to the product of their motivic symbols, provided at least one of the functions is completely multiplicative. The correspondence makes it very easy to prove many classical identities involving multiplicative functions.

From all of the above our main theorem follows: Classical multiplicative functions form a lambda-ring.

In future work, we intend to study the functor MS in more detail, and also generalize our main theorem to many other classes of multiplicative functions.

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Chapter 1

Introduction

In this paper we define motivic symbols, a new set-theoretical construction which are defined using multisets. To every commutative monoid one can associate a set of motivic symbols *with elements in that monoid*. Interestingly, the set of all motivic symbols with elements in a single commutative monoid will always admit a very well-behaved lambda-ring structure. Also, the method by which we associate a set of motivic symbols to a commutative monoid is a functor.

When we restrict our attention to sets of motivic symbols which are defined through commutative monoids that also admit a ring structure, with the monoid operation as multiplication, something particularly nice happens. More specifically, every motivic symbol will correspond to a unique sequence with elements in the ring which we are working in. This correspondence is important, because it preserves some interesting properties and greatly simplifies some otherwise complicated operations. It is possible to extend this notion to triples consisting of a commutative monoid, a commutative ring, and a monoid homomorphism from the monoid to the multiplicative structure of the ring.

We also present an important application of motivic symbols. We define a certain class of multiplicative function which we call *classical* multiplicative functions. Classical multiplicative functions are defined by specific properties of their Dirichlet series. The Bell series of a classical multiplicative function will always correspond uniquely to a motivic symbol, and the motivic symbol will look similar for the Bell series at all primes. This "similarity" can be made rigorous, and allow one to uniquely associate every classical multiplicative function to a motivic symbol with elements in a certain commutative monoid. This association allows one to equip the set of classical multiplicative functions with a lambda-ring structure. Also noteworthy, sum of motivic symbols in this monoid will correspond to Dirichlet convolution of classical multiplicative functions.

Every section begins with a short summary of its contents, but we will include a brief description here as well.

- The aim of chapter 2 is to define motivic symbols and introduce the functor MS from commutative monoids to lambda-rings, as well as the concept of corresponding sequences.
  - 2.1 Basic Definitions:
    We define basic concepts from set theory, abstract algebra, and category theory, and we lay out important conventions such as assuming commutativity.
  - 2.2 Constructing the MS-functor:
    We define motivic symbols, lambda-rings of motivic symbols, and the functor mapping commutative monoids and monoid homomorphisms to lambda-rings and lambda-ring homomorphisms.
  - 2.3 Properties of Motivic Symbols:
    We present smaller results that while mostly irrelevant for the main theorem, may still interesting in some other applications.
  - 2.4 Corresponding Sequences:
    We define the concept of sequences corresponding to motivic symbols and prove certain uniqueness and existence properties, as well as properties related to the lambda-ring operations on motivic symbols.
- In chapter 3, we precisely define what we mean by a multiplicative function being classical. We also state and prove the main theorem, which describes the lambda-ring structure on the set of classical multiplicative functions.
  - 3.1 Multiplicative Functions and their Generating series:
    We define various central concepts related to multiplicative functions, including the Bell series and Dirichlet series associated to a multiplicative function.
  - 3.2 Motivic symbols from Bell series:
    We explain how to construct a motivic symbol from a pair (f, p), where f is a multiplicative function, and p is a prime number.
  - 3.3 More Examples:
    We provide many examples of multiplicative functions, with data about their Bell series, Dirichlet series, and motivic symbols.
  - 3.4 Multiplicative Functions from Motivic Symbols:
    We define a certain monoid $\mathbb{M}$ and a certain lambda-ring $\mathbb{L}$. We also define the functions **mf** and **ds** from $\mathbb{L}$ to the set of multiplicative functions, and to the set of Dirichlet series, respectively.
  - 3.5 The Main Theorem:
    We define classical multiplicative functions and prove that there is a bijection between them and the lambda-ring $\mathbb{L}$. This induces a lambda-ring structure on the set of classical multiplicative functions, and we describe how this lambda-ring structure relates to Dirichlet convolution, multiplication of functions, and higher norm operators studied by Redmond and Sivaramakrishnan.

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Chapter 2

Motivic Symbols

2.1 Basic Definitions

The aim of this section is to define the basic concepts and conventions that are essential in this paper. Note in particular the conventions on monoids and rings in Subsections 2.1.1 and 2.1.2, and the notation for operations on multisets in Subsection 2.1.4.

2.1.1 Commutative Monoids

Definition 2.1.1. A commutative monoid $(S, \cdot)$ is a set S equipped with a binary operation $\cdot : S \times S \to S$ such that these axioms hold:

1. Associativity: Parentheses are not necessary.

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

   This allows us to write $a \cdot b \cdot c$, without ambiguity about what is computed first.

2. Commutativity: Order of computing doesn't matter.

$$a \cdot b = b \cdot a$$

3. Identity: There exists an element e, called the identity, such that:

$$a \cdot e = e \cdot a = a$$

Note 1. In this paper all monoids are commutative, unless explicitly stated otherwise.

Note 2. In this paper, $\mathbb{N}$ denotes the monoid of natural numbers under addition, which contains 0, unless explicitly stated otherwise. We do not explicitly use the monoid of natural numbers under multiplication.

Definition 2.1.2. A monoid homomorphism from a monoid $(M, \cdot)$ to a monoid $(N, \cdot)$ is a function $f : M \to N$ which satisfies these axioms:

1. $f(e) = e$

2. $f(a \cdot b) = f(a) \cdot f(b)$

Definition 2.1.3. An abelian group $(G, \cdot)$ is a commutative monoid in which for all $a \in G$, there exists an element $a^{-1} \in G$ with

$$a^{-1} \cdot a = e$$

where e is the identity element.

2.1.2 Commutative Rings

Definition 2.1.4. A commutative ring is a set S together with two binary operations $+, \cdot : S \times S \to S$, referred to as addition and multiplication respectively. Both addition and multiplication are required to satisfy the commutative monoid axioms, where we define 1 as the identity of multiplication and 0 as the identity of addition. The following axioms must also hold:

1. Inverse of Addition: For every element a, there exists an element $-a$ such that $-a + a = 0$.

2. Distributivity: Multiplication distributes over addition:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Definition 2.1.5. We define a ring homomorphism from the ring R to the ring S as a function between the underlying set, $f : R \to S$ for which the following axioms hold:

1. $f(1_R) = 1_S$, where 1_R is the multiplicative identity in the ring R.
2. $f(a + b) = f(a) + f(b)$
3. $f(a \cdot b) = f(a) \cdot f(b)$

Note 3. In this paper all rings are also commutative, unless explicitly stated otherwise.

Note 4. The only instance where $0 = 1$ is the zero ring, consisting of only one element.

Definition 2.1.6. An integral domain is a commutative ring in which the product of two non-zero elements is always non-zero.

2.1.3 Lambda-Rings

Definition 2.1.7. We define elementary symmetric polynomials $e_j(x_1, x_2, \ldots, x_r)$ by

$$\sum_{j=0}^{r} e_j(x_1, x_2, \ldots, x_r)t^j = \prod_{i=1}^{r}(1 + x_i t)$$

Definition 2.1.8. A lambda-rings is a commutative ring R together with operations $\lambda^k : R \to R$, for non-negative integers k, such that the following axioms hold for all $r, s \in R$:

1. $\lambda^0(r) = 1$

2. $\lambda^1 = id$

3. $\lambda^k(1) = 0$, for $k \geq 2$

4. $\lambda^k(r+s) = \sum_{n=0}^{k} \lambda^n(r)\lambda^{k-n}(s)$

5. $\lambda^k(rs) = P_k(\lambda^1(r), \ldots, \lambda^k(r), \lambda^1(s), \ldots, \lambda^k(s))$

6. $\lambda^n(\lambda^m(r)) = P_{n,m}(\lambda^1(r), \ldots, \lambda^{n \cdot m}(r))$

The polynomials P_k and $P_{n,m}$ are defined below. That these polynomials are well-defined, follows from the theory of symmetric polynomials.

Definition 2.1.9. Let $s_i = e_i(x_1, x_2, \ldots, x_r)$ and $\sigma_i = e_i(y_1, y_2, \ldots, y_r)$. Then we define the polynomial P_k of $2r$ variables by

$$\sum_{k=0}^{r} P_k(s_1, s_2, \ldots, s_r, \sigma_1, \sigma_2, \ldots, \sigma_r)t^k = \prod_{i=1}^{r}(1 + x_i y_i t)$$

Definition 2.1.10. Let $s_i = e_i(x_1, x_2, \ldots, x_r)$. Then we define the polynomial $P_{n,m}$ by

$$\sum_{k \geq 0} P_{n,m}(s_1, s_2, \ldots, s_{nm})t^k = \prod_{1 \leq i_1 < i_2 \ldots < i_m \leq nm}^{r} (1 + x_{i_1} x_{i_2} \ldots x_{i_m} t)$$

Definition 2.1.11. The Adams operations ψ^k for all integers $k \geq 1$ are operations on a lambda-ring R that are defined by the following identity for $x \in R$, $k \geq 1$ (The Newton Formula, see [1], Theorem 3.10):

$$\sum_{i=0}^{k-1}(-1)^i \lambda^i(x)\psi^{k-i}(x) = (-1)^{k+1} k \lambda^k(x)$$

Definition 2.1.12. A morphism of lambda-rings is defined as a morphism of the underlying commutative rings that also satisfies $f(\lambda^k(a)) = \lambda^k(f(a))$ for all k.

Definition 2.1.13. A psi-ring (or ψ-ring) is a commutative ring R together with psi-operations $\psi^k : R \to R$ for all integers $k \geq 1$. These operations satisfy the following axioms:

1. ψ^k are all ring endomorphisms (a ring homomorphism from a ring to itself). This means:
   (a) $\psi^k(1) = 1$
   (b) $\psi^k(r+s) = \psi^k(r) + \psi^k(s)$
   (c) $\psi^k(r \cdot s) = \psi^k(r) \cdot \psi^k(s)$

2. $\psi^1 = id$

3. $\psi^n \circ \psi^m = \psi^{nm}$

Theorem 2.1.1 (Wilkerson's theorem)**.** Let R be a $\mathbb{Z}$-torsion-free psi-ring in which

$$\psi^p(x) \equiv x^p \pmod{pR}$$

for each $x \in R$ and prime p. Then there exists a unique lambda-ring structure on R whose Adams operations are the psi-operations of R.

Proof. See Theorem 3.54 in [1]. □

This theorem will be used to prove that our construction is actually a lambda-ring.

Theorem 2.1.2 (Corollary 3.16 in [1])**.** Let $f : R \to S$ be a ring homomorphism between two lambda-rings in which S is torsion-free. Suppose that $f \circ \psi^n = \psi^n \circ f$ for all $n \geq 1$. Then f is a lambda-ring homomorphism.

Proof. See Corollary 3.16 in [1]. □

2.1.4 Multisets

In this subsection we define central terms surrounding multisets, and prove various results about them.

Definition 2.1.14. A multiset is a set where multiple instances of the same element are allowed.

Note 5. All multisets in this paper are assumed to be finite, meaning the amount of all elements, counted with multiplicity, is finite.

Definition 2.1.15. Multisets have a multiplicity function from a set containing all the unique elements of the multiset to the non-negative integers. This function describes how many of an object are in the multiset. Normal notation for this function is 1_A, where A is a multiset.

Example 2.1.1. $\{1, 1, 2, 5, 5, 5\}$ is a multiset. It's multiplicity function maps $1 \mapsto 2$, $2 \mapsto 1$, $5 \mapsto 3$.

Definition 2.1.16. The disjoint union $A \uplus B$ represents pointwise addition of the multiplicity functions A and B. In other words, whether or not elements are distinct is irrelevant.

Example 2.1.2. $\{a, a\} \uplus \{a, b\} = \{a, a, a, b\}$.

Definition 2.1.17. Let $(M, \cdot)$ be a monoid, and let A and B be multisets with elements in M. We define the monoid product of A and B, $A \bullet B$, as the set of all products one can make with an element from A and an element from B. In terms of multiplicity functions:

$$1_{A \bullet B}(x) = \sum_{\substack{r,s \in M \\ r \cdot s = x}} 1_A(r) \cdot 1_B(s)$$

Example 2.1.3. $\{a, a\} \bullet \{a, b\} = \{a \cdot a, a \cdot b, a \cdot a, a \cdot b\}$.

Definition 2.1.18. The intersection of two multisets A and B, written $A \cap B$, is defined such that $1_{A \cap B}(x) = min(1_A(x), 1_B(x))$.

Example 2.1.4.

$$\{1,1,1,2\} \cap \{1,1,3\} = \{1,1\}$$

Definition 2.1.19. For two multisets A and B, the multiset difference $A \backslash B$ is defined such that $1_{A \backslash B}(x) = max(0, 1_A(x) - 1_B(x))$.

Example 2.1.5.

$$\{1,1,1,2\} \backslash \{1,1,3\} = \{1,2\}$$

Definition 2.1.20. The image of a multiset A under a function f, written $f(A)$, is defined as the multiset one gets by passing every element of A through f. In terms of multiplicity functions:

$$1_{f(A)}(x) = \sum_{\substack{r \in M \\ f(r)=x}} 1_A(r)$$

where M is some set that contains all elements of A.

Example 2.1.6. Let $f : x \mapsto x^2$. Then we have

$$f(\{1,-1,4,4\}) = \{1,1,16,16\}$$

Proposition 2.1.3. Let A, B be multisets with elements in the monoid $(M, \cdot)$. Then $A \bullet B = B \bullet A$.

Proof. This follows trivially from the commutativity of a monoid. Let's show a proof using multiplicity functions as well though. We want to show $A \bullet B = B \bullet A$. Using multiplicity functions, this is equivalent to

$$1_{A \bullet B}(x) = 1_{B \bullet A}(x)$$

Which is by definition equivalent to

$$\sum_{\substack{r,s \in M \\ r \cdot s = x}} 1_A(r) \cdot 1_B(s) = \sum_{\substack{r,s \in M \\ r \cdot s = x}} 1_B(r) \cdot 1_A(s)$$

On the left-hand side, swap r and s everywhere. Since $\cdot$ is commutative, $r \cdot s = x \Leftrightarrow s \cdot r = x$. Pointwise product of functions with codomain $\mathbb{N}$ is commutative, so the two sides are equal. □

Proposition 2.1.4. Let A, B, C be multisets with elements in the monoid $(M, \cdot)$. Then $(A \bullet B) \bullet C = A \bullet (B \bullet C)$.

Proof. This follows trivially from the associativity of a monoid. Let's prove this using multiplicity functions as well. We want to prove that $A \bullet (B \bullet C) = (A \bullet B) \bullet C$. This is equivalent to $1_{A \bullet (B \bullet C)} = 1_{(A \bullet B) \bullet C}$, which again is by definition equivalent to

$$\sum_{\substack{r,s \in M \\ r \cdot s = x}} 1_A(r) \cdot 1_{B \bullet C}(s) = \sum_{\substack{r,s \in M \\ r \cdot s = x}} 1_{A \bullet B}(r) \cdot 1_C(s)$$

$$\sum_{\substack{r,s \in M \\ r \cdot s = x}} 1_A(r) \cdot \sum_{\substack{t,u \in M \\ t \cdot u = s}} 1_B(t) \cdot 1_C(u) = \sum_{\substack{r,s \in M \\ r \cdot s = x}} \left(\sum_{\substack{t,u \in M \\ t \cdot u = r}} 1_A(t) \cdot 1_B(u) \right) \cdot 1_C(s)$$

$$\sum_{\substack{r,s\in M\\ r\cdot s=x}}\sum_{\substack{t,u\in M\\ t\cdot u=s}} 1_A(r)\cdot 1_B(t)\cdot 1_C(u) = \sum_{\substack{r,s\in M\\ r\cdot s=x}}\sum_{\substack{t,u\in M\\ t\cdot u=r}} 1_A(t)\cdot 1_B(u)\cdot 1_C(s)$$

$$\sum_{\substack{r,t,u\in M\\ r\cdot t\cdot u=x}} 1_A(r)\cdot 1_B(t)\cdot 1_C(u) = \sum_{\substack{t,u,s\in M\\ t\cdot u\cdot s=x}} 1_A(t)\cdot 1_B(u)\cdot 1_C(s)$$

The two are clearly equal, the only difference between the expressions are the choice of letters. □

Proposition 2.1.5. Let A, B, C be multisets with elements in the monoid $(M,\cdot)$. The operation on multisets $\bullet$ distributes over $\uplus$.

Proof. This follows trivially from basic arithmetic. Let's present a proof using multiplicity functions as well. The statement can be written as $A\bullet(B\uplus C) = A\bullet B\uplus A\bullet C$. Let's once again use arithmetic on multiplicity functions. We have to prove that:

$$1_{A\bullet(B\uplus C)}(x) = 1_{A\bullet B\uplus A\bullet C}(x)$$

which is equivalent to

$$\sum_{\substack{r,s\in M\\ r\cdot s=x}} 1_A(r)\cdot 1_{B\uplus C}(s) = 1_{A\bullet B}(x) + 1_{A\bullet C}(x)$$

$$\sum_{\substack{r,s\in M\\ r\cdot s=x}} 1_A(r)\cdot(1_B(s)+1_C(s)) = \sum_{\substack{r,s\in M\\ r\cdot s=x}} 1_A(r)\cdot 1_B(s) + \sum_{\substack{r,s\in M\\ r\cdot s=x}} 1_A(r)\cdot 1_C(s)$$

$$\sum_{\substack{r,s\in M\\ r\cdot s=x}} 1_A(r)\cdot 1_B(s) + 1_A(r)\cdot 1_C(s) = \sum_{\substack{r,s\in M\\ r\cdot s=x}} 1_A(r)\cdot 1_B(s) + \sum_{\substack{r,s\in M\\ r\cdot s=x}} 1_A(r)\cdot 1_C(s)$$

We can expand this to

$$\sum_{\substack{r,s\in M\\ r\cdot s=x}} 1_A(r)\cdot 1_B(s) + \sum_{\substack{r,s\in M\\ r\cdot s=x}} 1_A(r)\cdot 1_C(s) = \sum_{\substack{r,s\in M\\ r\cdot s=x}} 1_A(r)\cdot 1_B(s) + \sum_{\substack{r,s\in M\\ r\cdot s=x}} 1_A(r)\cdot 1_C(s)$$

□

Proposition 2.1.6. Let A, B, C be multisets with elements in the monoid $(M,\cdot)$. Taking the image of a disjoint union $A\uplus B$ under a function f, is the same as taking the disjoint union $\uplus$ of the images of A and B under the function f. In other words, $f(A\uplus B) = f(A)\uplus f(B)$.

Proof. This also follows trivially from basic arithmetic. Let's present a proof using multiplicity functions as well. In terms of multiplicity functions, we want to prove $1_{f(A\uplus B)} = 1_{f(A)\uplus f(B)}$. Using our definitions:

$$1_{f(A)} + 1_{f(B)} = \sum_{\substack{r\in M\\ f(r)=x}} 1_{A\uplus B}(r)$$

$$\sum_{\substack{r\in M\\ f(r)=x}} 1_A(r) + \sum_{\substack{r\in M\\ f(r)=x}} 1_B(r) = \sum_{\substack{r\in M\\ f(r)=x}} 1_A(r) + 1_B(r)$$

This follows from the fact that we expand $\sum a+b$ into $\sum a + \sum b$. □

Proposition 2.1.7. Assume we are working in a monoid $(M, \cdot)$. Taking the image of a multiset product $A \bullet B$ under a monoid homomorphism f, is the same as taking the multiset product $\bullet$ of the images of A and B under f. In other words, $f(A \bullet B) = f(A) \bullet f(B)$.

Proof. This follows trivially from the axioms of a monoid homomorphism. Let's prove this using multiplicity functions as well. We want to prove $1_{f(A \bullet B)} = 1_{f(A) \bullet f(B)}$. We can start by using our definitions:

$$\sum_{\substack{u \in M \\ f(u)=x}} 1_{A \bullet B}(u) = \sum_{\substack{r,s \in M \\ r \cdot s = x}} 1_{f(A)}(r) \cdot 1_{f(B)}(s)$$

$$\sum_{\substack{u \in M \\ f(u)=x}} \sum_{\substack{c,d \in M \\ c \cdot d = u}} 1_A(c) \cdot 1_B(d) = \sum_{\substack{r,s \in M \\ r \cdot s = x}} \left(\sum_{\substack{u \in M \\ f(u)=r}} 1_A(u) \right) \cdot \left(\sum_{\substack{v \in M \\ f(v)=s}} 1_B(v) \right)$$

By the expanding the product on the right side, we get

$$\sum_{\substack{u \in M \\ f(u)=x}} \sum_{\substack{r,s \in M \\ r \cdot s = u}} 1_A(r) \cdot 1_B(s) = \sum_{\substack{r,s \in M \\ r \cdot s = x}} \sum_{\substack{u,v \in M \\ f(u)=r \\ f(v)=s}} 1_A(u) \cdot 1_B(v)$$

We can combine the sum conditions in the following way:

$$\sum_{\substack{r,s \in M \\ f(r \cdot s)=x}} 1_A(r) \cdot 1_B(s) = \sum_{\substack{u,v \in M \\ f(u) \cdot f(v)=x}} 1_A(u) \cdot 1_B(v)$$

This is given by $f(a \cdot b) = f(a) \cdot f(b)$, which is an axiom for monoid homomorphisms. The choice of letters doesn't matter. □

2.1.5 Basic Category Theory and Functors

Definition 2.1.21. A category $\mathscr{C}$ consists of three parts:

1. $Ob(\mathscr{C})$, a class whose elements are called the objects in the category. In many interesting cases, these objects are algebraic structures.

2. For every pair of objects (A, B), a class $Hom_{\mathscr{C}}(A, B)$ or $Hom(A, B)$ called the morphisms from A to B. In the case of objects being algebraic structures, morphisms are often homomorphisms. If $f \in Hom(A, B)$ we can write $f : A \to B$

3. A binary operation $\circ : Hom(B, C) \times Hom(A, B) \to Hom(A, C)$ called the composition operation. This operation satisfies these axioms:

   (a) Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$, where $f : A \to B$, $g : B \to C$, $f : C \to D$

   (b) Identity: For every object A, there exists a morphism $id_A : A \to A$, referred to as the identity morphism of A such that for every morphism $f : A \to B$, $id_B \circ f = f = f \circ id_A$.

Definition 2.1.22. A (covariant) functor F from the category $\mathscr{C}$ to the category $\mathscr{D}$ is a function $F : Ob(\mathscr{C}) \to Ob(\mathscr{D})$ and functions $F : Hom_{\mathscr{C}}(A, B) \to Hom_{\mathscr{D}}(F(A), F(B))$ for all objects A and B such that the following properties hold:

1. For all objects A, $F(id_A) = id_{F(A)}$.
2. For all morphisms $f : A \to B$ and $g : B \to C$, $F(g \circ f) = F(g) \circ F(f)$

2.2 Constructing the MS-functor

The aim of this section is to define motivic symbols, basic properties of them, various operations on them, and finally the functor MS. We also prove that MS actually is a functor.

2.2.1 Definition of Motivic Symbols

In this subsection we define motivic symbols, superdimension, and motivic pre-symbols.

Definition 2.2.1. A motivic symbol is an ordered pair of disjoint finite multisets with elements in a monoid M. A motivic symbol is written $\frac{A}{B}$ or A/B, where A and B are the multisets. Here, A is called the upstairs, and B is called the downstairs part of the motivic symbol.

Definition 2.2.2. The superdimension of a motivic symbol is an ordered pair (a, b), where a is the cardinality of A and b is the cardinality of B.

Definition 2.2.3. A motivic pre-symbol is the same as a motivic symbol, except that the multisets are not required to be disjoint.

2.2.2 Operations

In this subsection, we define the direct sum, tensor product, and the Adams operations. We also outline and prove some basic properties of those operations.

Note 6. There are many operations one can define on the set of motivic symbols with elements in the monoid M. The definition above mentions that A and B must be disjoint. Many of these operations may output motivic symbols whose upstairs and downstairs have common elements, i.e. a motivic pre-symbol. To fix this, we include cancellation of any common elements in all operations, functions, etc. . .

Definition 2.2.4. Cancellation means that if we have a motivic pre-symbol A/B, we can construct a motivic symbol. This is done by taking the intersection C of A and B, and the letting the new motivic symbol equal $(A\backslash C)/(B\backslash C)$.

Example 2.2.1. Here are two examples of cancellation:

$$\frac{\{1,4,2,2\}}{\{1,4,3,2\}} \to \frac{\{2\}}{\{3\}}, \frac{\{3,2,1\}}{\{1,3,7,5\}} \to \frac{\{2\}}{\{7,5\}}$$

Definition 2.2.5. The direct sum of S_0 and S_1, written $S_0 \oplus S_1$, is defined by

$$\frac{A}{B} \oplus \frac{C}{D} := \frac{A \uplus C}{B \uplus D}$$

with cancellation of any common elements in the result.

Definition 2.2.6. The tensor product of S_0 and S_1, written $S_0 \otimes S_1$, is defined by

$$\frac{A}{B} \otimes \frac{C}{D} := \frac{A \bullet C \uplus B \bullet D}{A \bullet D \uplus B \bullet C} = \frac{A \bullet C}{A \bullet D} \oplus \frac{B \bullet D}{B \bullet C}$$

also with cancellation of any common elements in the result.

Definition 2.2.7. The Adams operations are defined by

$$\psi^n\left(\frac{\{a_1, a_2, a_3, \dots, a_k\}}{\{b_1, b_2, b_3, \dots, b_m\}}\right) := \frac{\{a_1^n, a_2^n, a_3^n, \dots a_k^n\}}{\{b_1^n, b_2^n, b_3^n, \dots, b_m^n\}}$$

also with cancellation of common elements. Here, a^n refers to the monoid product of a with itself n times.

Example 2.2.2. Consider motivic symbols with elements in the monoid $\mathbb{Z}$ under multiplication.

1. Direct sum:
$$\frac{\{2,4\}}{\{1\}} \oplus \frac{\{1,4,6,90\}}{\{2,2\}} = \frac{\{2,4,1,4,6,90\}}{\{1,2,2\}} = \frac{\{4,4,6,90\}}{\{2\}}$$

2. Tensor product:
$$\frac{\{2,1\}}{\{3\}} \otimes \frac{\{3,2\}}{\{2\}} = \frac{\{6,4,3,2,6\}}{\{4,2,9,6\}} = \frac{\{6,3\}}{\{9\}}$$

3. An Adams operation:
$$\psi^2\left(\frac{\{1,4\}}{\{-1,3,6,2\}}\right) = \frac{\{1^2,4^2\}}{\{(-1)^2,3^2,6^2,2^2\}} = \frac{\{1,16\}}{\{1,9,36,4\}} = \frac{\{16\}}{\{9,36,4\}}$$

Proposition 2.2.1. Both direct sum and tensor product are commutative.

Proof.

1. Commutativity of direct sum is given for free by the fact that $\uplus$ is commutative.

2. Commutativity of the tensor product can be formulated as
$$\frac{A \bullet C \uplus B \bullet D}{A \bullet D \uplus B \bullet C} = \frac{C \bullet A \uplus D \bullet B}{C \bullet B \uplus D \bullet A}$$
Since we know that $\uplus$ is commutative, it is sufficient to prove that
$$\frac{A \bullet C \uplus B \bullet D}{A \bullet D \uplus B \bullet C} = \frac{C \bullet A \uplus D \bullet B}{D \bullet A \uplus C \bullet B}$$
This follows from the fact that the operation $\bullet$ on multisets is commutative (see Proposition 2.1.3). □

Proposition 2.2.2. Both direct sum and tensor product are associative.

Proof.

1. Associativity of direct sum is given for free by the fact that $\uplus$ is associative.
2. Associativity of tensor product can be formulated as

$$\frac{A}{B} \otimes \left(\frac{C}{D} \otimes \frac{E}{F}\right) = \left(\frac{A}{B} \otimes \frac{C}{D}\right) \otimes \frac{E}{F}$$

Let's use the definition of $\otimes$ (see Definition 2.2.6) to expand this.

$$\frac{A}{B} \otimes \frac{C \bullet E \uplus D \bullet F}{C \bullet F \uplus D \bullet E} = \frac{A \bullet C \uplus B \bullet D}{A \bullet D \uplus B \bullet C} \otimes \frac{E}{F}$$

$$\frac{A}{B} \otimes \frac{(C \bullet E \uplus D \bullet F)}{(C \bullet F \uplus D \bullet E)} = \frac{(A \bullet C \uplus B \bullet D)}{(A \bullet D \uplus B \bullet C)} \otimes \frac{E}{F}$$

$$\frac{A \bullet (C \bullet E \uplus D \bullet F) \uplus B \bullet (C \bullet F \uplus D \bullet E)}{A \bullet (C \bullet F \uplus D \bullet E) \uplus B \bullet (C \bullet E \uplus D \bullet F)}$$

$$= \frac{(A \bullet C \uplus B \bullet D) \bullet E \uplus (A \bullet D \uplus B \bullet C) \bullet F}{(A \bullet C \uplus B \bullet D) \bullet F \uplus (A \bullet D \uplus B \bullet C) \bullet E}$$

From Proposition 2.1.5, we have that $A \bullet (B \uplus C) = A \bullet B \uplus A \bullet C$. Putting this into our equation, we get:

$$\frac{A \bullet (C \bullet E) \uplus A \bullet (D \bullet F) \uplus B \bullet (C \bullet F) \uplus B \bullet (D \bullet E)}{A \bullet (C \bullet F) \uplus A \bullet (D \bullet E) \uplus B \bullet (C \bullet E) \uplus B \bullet (D \bullet F)}$$

$$= \frac{(A \bullet C) \bullet E \uplus (B \bullet D) \bullet E \uplus (A \bullet D) \bullet F \uplus (B \bullet C) \bullet F}{(A \bullet C) \bullet F \uplus (B \bullet D) \bullet F \uplus (A \bullet D) \bullet E \uplus (B \bullet C) \bullet E}$$

Let's change the order of the right-hand side a bit:

$$\frac{A \bullet (C \bullet E) \uplus A \bullet (D \bullet F) \uplus B \bullet (C \bullet F) \uplus B \bullet (D \bullet E)}{A \bullet (C \bullet F) \uplus A \bullet (D \bullet E) \uplus B \bullet (C \bullet E) \uplus B \bullet (D \bullet F)}$$

$$= \frac{(A \bullet C) \bullet E \uplus (A \bullet D) \bullet F \uplus (B \bullet C) \bullet F \uplus (B \bullet D) \bullet E}{(A \bullet C) \bullet F \uplus (A \bullet D) \bullet E \uplus (B \bullet C) \bullet E \uplus (B \bullet D) \bullet F}$$

It is clear that this is true if the operation $\bullet$ on multisets is associative. This is proven in Proposition 2.1.4. □

Proposition 2.2.3. The tensor product distributes over direct sum, i.e. $S_0 \otimes (S_1 \oplus S_2) = S_0 \otimes S_1 \oplus S_0 \otimes S_2$.

Proof. We need to prove that

$$\frac{A}{B} \otimes \left(\frac{C}{D} \oplus \frac{E}{F}\right) = \frac{A}{B} \otimes \frac{C}{D} \oplus \frac{A}{B} \otimes \frac{E}{F}$$

Let's rewrite the left-hand side using the definitions of tensor product and direct sum (see Definitions 2.2.6 and 2.2.5):

$$\frac{A}{B} \otimes \left(\frac{C}{D} \oplus \frac{E}{F}\right) = \frac{A}{B} \otimes \frac{C \uplus E}{D \uplus F}$$

$$= \frac{A \bullet (C \uplus E) \uplus B \bullet (D \uplus F)}{A \bullet (D \uplus F) \uplus B \bullet (C \uplus E)}$$

Using Proposition 2.1.5, we get:

$$\frac{A \bullet C \uplus A \bullet E \uplus B \bullet D \uplus B \bullet F}{A \bullet D \uplus A \bullet F \uplus B \bullet C \uplus B \bullet E}$$

Now let's expand the right-hand side.

$$\frac{A}{B} \otimes \frac{C}{D} \oplus \frac{A}{B} \otimes \frac{E}{F}$$

$$= \frac{A \bullet C \uplus B \bullet D}{A \bullet D \uplus B \bullet C} \oplus \frac{A \bullet E \uplus B \bullet F}{A \bullet F \uplus B \bullet E}$$

$$= \frac{A \bullet C \uplus B \bullet D \uplus A \bullet E \uplus B \bullet F}{A \bullet D \uplus B \bullet C \uplus A \bullet F \uplus B \bullet E}$$

$$= \frac{A \bullet C \uplus A \bullet E \uplus B \bullet D \uplus B \bullet F}{A \bullet D \uplus A \bullet F \uplus B \bullet C \uplus B \bullet E}$$

Thus, the right-hand side and left-hand side are equal. □

Proposition 2.2.4.

1. The only identity element of the direct sum is $\varnothing/\varnothing$.
2. The only identity of the tensor product is $\{1\}/\varnothing$, where 1 is the identity in the monoid we are working with.

Proof.

1. $\varnothing/\varnothing$ works as an identity:

$$\frac{A}{B} \oplus \frac{\varnothing}{\varnothing} = \frac{A \uplus \varnothing}{B \uplus \varnothing} = \frac{A}{B}$$

2. $\{1\}/\varnothing$ works as an identity:

$$\frac{A}{B} \otimes \frac{\{1\}}{\varnothing} = \frac{A \bullet \{1\}}{B \bullet \{1\}} = \frac{A}{B}$$

To show that the identities are unique, we prove that all identities are equal. This holds for all commutative binary operations, including tensor product and direct sum. Assume e_0 and e_1 are identities. Then, by definition, $e_0 = e_0 \cdot e_1 = e_1$. □

Proposition 2.2.5. It is possible to construct additive inverses simply by swapping the upstairs and the downstairs.

Proof. This works because of cancellation,

$$\frac{A}{B} \oplus \frac{B}{A} = \frac{A \uplus B}{A \uplus B} = \frac{\varnothing}{\varnothing}$$

□

Definition 2.2.8. We denote the additive inverse by $\ominus A/B = B/A$.
$A/B \ominus C/D$ is shorthand for $A/B \oplus \ominus C/D$.

Definition 2.2.9. We also define $n \cdot S$, where n is an integer, as $S \oplus S \oplus S \oplus \dots$ n times if n is positive, $\varnothing/\varnothing$ if $n = 0$, and $\ominus(S \oplus S \oplus S \oplus \dots)$ $|n|$ times if n is negative.

Definition 2.2.10. Finally, $\frac{A}{B}^{\otimes n}$, for non-negative integers n, is defined as $\frac{\{1\}}{\varnothing}$ if $n = 0$, and the tensor product of A/B with itself n times otherwise.

2.2.3 The Lambda-Ring Structure

The goal of this subsection is to prove that given a monoid, the set of motivic symbols with elements in that monoid form a lambda-ring with the operations defined in Subsection 2.2.2. Also, we define the term "MS-ring".

Definition 2.2.11. Given a monoid M we define $MS(M)$ to be set of all motivic symbols with elements in M. We equip this set with addition which is direct sum, multiplication, which is the tensor product, and the Adams operations, which are defined as ψ^n.

Theorem 2.2.6. If M is a monoid, $MS(M)$ is a lambda-ring.

Proof. It is clear that $MS(M)$ is closed under direct sum, tensor product, and the Adams operations. By Propositions 2.2.2, 2.2.1, and 2.2.4, as well as Propositions 2.2.5 and 2.2.3, we get that $MS(M)$ is a commutative ring for all commutative monoids M.

However, as mentioned, $MS(M)$ is not only a commutative ring, but a lambda-ring. We need to prove that ψ^n are psi-operations, and that they induce a lambda-ring structure.

1. ψ^k are all ring endomorphisms.
   (a) They clearly map $\{1\}/\varnothing$ to $\{1^n\}/\varnothing = \{1\}/\varnothing$.
   (b) Direct sum is just union of the upstairs and downstairs, so clearly, $\psi^n(S_0 \oplus S_1) = \psi^n(S_0) \oplus \psi^n(S_1)$. (see Proposition 2.1.6)
   (c) Tensor product is harder. The statement we want to prove is

   $$\psi^n\left(\frac{A}{B} \otimes \frac{C}{D}\right) = \psi^n\left(\frac{A}{B}\right) \otimes \psi^n\left(\frac{C}{D}\right)$$

   Let $\psi^n(A)$, where A is a multiset, denote the multiset given by raising every element of A to the power of n. Using this, we can rewrite the right-hand side as

   $$\frac{\psi^n(A)}{\psi^n(B)} \otimes \frac{\psi^n(C)}{\psi^n(D)}$$

   Let's expand this side using the definition of tensor product. (see Definition 2.2.6)

   $$\frac{\psi^n(A) \cdot \psi^n(C) \uplus \psi^n(B) \cdot \psi^n(D)}{\psi^n(A) \cdot \psi^n(D) \uplus \psi^n(B) \cdot \psi^n(C)}$$

Let's look at a single term of the form $\psi^n(A) \bullet \psi^n(B)$. We want to show that this equals $\psi^n(A \bullet B)$. Note that ψ^n of a multiset can be thought of as the image of A under the function mapping an element a to a^n. Let's name this function $-^n$. $-^n$ is a monoid homomorphism (since $a^n b^n = (ab)^n$), so we can use Proposition 2.1.7. Now, let's put this equality into our expression.

$$\frac{\psi^n(A \bullet C) \uplus \psi^n(B \bullet D)}{\psi^n(A \bullet D) \uplus \psi^n(B \bullet C)}$$

Because of Proposition 2.1.6, which states that $f(A \uplus B) = f(A) \uplus f(B)$ for a function f and multisets A and B, the above is equal to

$$\frac{\psi^n(A \bullet C \uplus B \bullet D)}{\psi^n(A \bullet D \uplus B \bullet C)}$$

By the definition of the tensor product, this equals

$$\psi^n\left(\frac{A}{B} \otimes \frac{C}{D}\right)$$

2. $\psi^1 = id$. This clearly true, due to $m^1 = m$ for any element $m \in M$.
3. $\psi^n \circ \psi^m = \psi^{nm}$. This follows from the definition of ψ^n and the fact that $(r^m)^n = r^{n \cdot m}$ for any $r \in M$.

In order to prove that these ψ-operations induce a lambda-ring structure on $MS(M)$, we can use Wilkerson's theorem (Theorem 2.1.1). First, we need to prove that $MS(M)$ is torsion-free, meaning that repeated addition of a non-zero motivic symbol never results in zero. In the language of motivic symbols,

Lemma 2.2.7. for all $n \geq 1$,

$$n \cdot \frac{A}{B} = \frac{\varnothing}{\varnothing} \Rightarrow \frac{A}{B} = \frac{\varnothing}{\varnothing}$$

Proof. By definition, $n \cdot A/B$ can be written

$$\frac{A \uplus A \dots}{B \uplus B \dots}$$

with n copies upstairs and downstairs. Since A/B is a motivic symbol, A and B have no common elements. The same will therefore hold for $A \uplus A \dots$ and $B \uplus B \dots$ as well. This means that if the superdimension of A/B is (a, b), the super dimension of $n \cdot A/B$ will be $(n \cdot a, n \cdot b)$. We have $n \cdot A/B = \varnothing/\varnothing$, so $(n \cdot a, n \cdot b) = (0, 0)$. $n \neq 0$, so $(a, b) = (0, 0)$, which again implies that $A/B = \varnothing/\varnothing$. □

The other condition of Wilkerson's theorem (Theorem 2.1.1) requires $\psi^p(x) \equiv x^p \mod pR$ for all prime natural numbers p. When that is proven, we know that there exists a unique lambda-ring structure on $MS(M)$ whose Adams operations are ψ^n. Let's first translate the condition to the language of motivic symbols:

$$\psi^p\left(\frac{A}{B}\right) \equiv \left(\frac{A}{B}\right)^{\bigotimes p} \pmod{pR} \qquad (*)$$

We know that A/B can be rewritten as

$$\frac{A}{B} = \left(\bigoplus_{a\in A} \frac{\{a\}}{\varnothing}\right) \oplus \left(\bigoplus_{b\in B} \frac{\varnothing}{\{b\}}\right)$$

So, rewriting the left-hand side of $(*)$ again gives us

$$\psi^p\left(\left(\bigoplus_{a\in A} \frac{\{a\}}{\varnothing}\right) \oplus \left(\bigoplus_{b\in B} \frac{\varnothing}{\{b\}}\right)\right) = \left(\bigoplus_{a\in A} \frac{\{a^p\}}{\varnothing}\right) \oplus \left(\bigoplus_{b\in B} \frac{\varnothing}{\{b^p\}}\right)$$

The right-hand side is a bit trickier. However, we know that the tensor product is distributive over direct sum, so we can use what we know about similar statements in elementary algebra to rewrite. We know from elementary algebra, more specifically the multinomial theorem (see [7]), that

$$(a_0 + a_1 + \ldots + a_j)^n = \sum_{k_1+k_2+\ldots+k_j=n} \binom{n}{k_1, k_2, \ldots, k_j} \prod_{t=1}^{j} a_t^{k_t}$$

where $\binom{n}{k_1,k_2,\ldots,k_j}$ is the multinomial coefficient, defined as:

$$\binom{n}{k_1, k_2, \ldots, k_j} = \frac{n!}{k_1!, k_2!, \ldots, k_j!}$$

Then, by indexing the terms of A as $\{a_t\}$ and the one of B as $\{b_t\}$, we can rewrite the right-hand side as

$$\bigoplus_{\sum n_i + \sum m_i = p} \binom{p}{n_1, n_2, \ldots, n_j, m_1, m_2, \ldots, m_j} \cdot \bigotimes_{t=1}^{j} \frac{\{a_t^{n_t}\}}{\varnothing} \otimes \bigotimes_{t=1}^{j} \frac{\varnothing}{\{b_t^{m_t}\}}$$

Let's focus on the multinomial coefficient, namely

$$\binom{p}{n_1, n_2, \ldots, n_j, m_1, m_2, \ldots, m_j} = \frac{p!}{n_1!, n_2!, \ldots, n_j!, m_1!, m_2!, \ldots, m_j!}$$

It is apparent that all terms have this integer factor for some values of $n_1 + n_2 + \ldots + n_j + m_1 + m_2 + \ldots + m_j = p$. Obviously, the upstairs contains a factor of p. This factor can only disappear is if any of the elements $n_1, n_2, \ldots, n_j, m_1, m_2, \ldots, m_j$ has any factors of p, except 1 of course. Since p is a prime, the only way p can be canceled is thus if one of the elements $n_1, n_2, \ldots, n_j, m_1, m_2, \ldots, m_j$ contain p as a factor, and must thus equal p, since the sum of all must be p and none can be negative. This also implies that all other elements equal 0, and furthermore that the multinomial coefficient equals 1. This means that for p not to be a factor, the term can be written either

$$\frac{\{a_t^p\}}{\varnothing} \quad \text{or} \quad \frac{\varnothing}{\{b_t^p\}}$$

Note that there will be exactly one of these for every a_t and b_t. Now back to $(*)$. Everything that has a factor of p, is canceled because we are working modulo p. Thus, the only terms that remain on the right side are the terms above, of which there are one for each element. Thus the right-hand side can be rewritten modulo p as:

$$\left(\bigoplus_{a\in A}\frac{\{a^p\}}{\varnothing}\right)\oplus\left(\bigoplus_{b\in B}\frac{\varnothing}{\{b^p\}}\right)$$

Note that this is equal to the left-hand side, and hence we have proved that the condition $(*)$ is satisfied in $MS(M)$. □

Definition 2.2.12. An MS-Ring is a lambda-ring of the form $MS(M)$, for some monoid M.

Example 2.2.3. The simplest example of an MS-Ring is the one given by the monoid consisting of only one element. This lambda-ring is isomorphic to $\mathbb{Z}$, with the canonical ring structure and id as Adams operations for all n. This can be proven by mapping a motivic symbol to the number of elements in the upstairs part, minus the number of elements in the downstairs part. Note that all motivic symbols in this ring have a superdimension of either $(a, 0)$ or $(0, a)$, for some a.

2.2.4 Functoriality

In this subsection, we first define some categories, then we define the functor MS, and finally, prove that it actually is a functor.

Definition 2.2.13. The category of commutative monoids, denoted here by $CMon$, has commutative monoids as objects, monoid homomorphisms as morphisms, and standard function composition as the composition operation.

Definition 2.2.14. The category of commutative rings, *CRing* has commutative rings as objects, ring homomorphisms as morphisms, and again, standard function composition as the composition operation.

Definition 2.2.15. The category of lambda-rings, $Ring^\lambda$, has lambda rings as objects, lambda-ring homomorphisms as morphisms, and once again, standard function composition as the composition operation.

Definition 2.2.16. We want to define MS as a covariant functor from $CMon$ to $Ring^\lambda$, which maps a commutative monoid M to the associated lambda ring $MS(M)$. In summary:

1. The underlying set of $MS(M)$ is the set of motivic symbols with elements in M.
2. Addition in $MS(M)$ is the direct sum $\oplus$, defined in Definition 2.2.5.
3. Multiplication in $MS(M)$ is the tensor product $\otimes$, defined in Definition 2.2.6.
4. The lambda operations in $MS(M)$ are defined through the Adams operations ψ^n. Those are defined in Definition 2.2.7.

It maps morphisms in the following manner:

$$MS : f \mapsto \left(\frac{A}{B} \mapsto \frac{f(A)}{f(B)} \right)$$

where $f(A)$ refers to the image of the multiset A under the function f. In other words, a monoid homomorphism is mapped to the lambda-ring homomorphism that runs all the elements in the motivic symbol through the monoid homomorphism.

Remark 1. Sometimes, we want to consider motivic symbols with elements in a commutative ring, rather than a monoid. In these cases, we forget the additive structure, which leaves a monoid. Writing $MS(R)$, where R is a ring, is shorthand for the MS-Ring generated by the multiplicative structure of R.

Theorem 2.2.8. *MS is a covariant functor.*

Proof. To prove that MS is a functor, we first need to prove that for all monoid homomorphisms f, $MS(f)$ is a lambda-ring homomorphism.

Let's go through the axioms of a lambda-ring homomorphism.

1. Lambda-ring homomorphisms must map the identity to the identity. In other words, the following must hold:

$$MS(f) \left(\frac{\{1\}}{\varnothing} \right) = \frac{\{1\}}{\varnothing}$$

By the definition of $MS(f)$, this is equivalent to

$$\frac{\{f(1)\}}{\varnothing} = \frac{\{1\}}{\varnothing}$$

We know by the monoid homomorphism axioms that $f(1) = 1$.

2. $MS(f)$ should map direct sums to direct sums. That is, we need

$$MS(f) \left(\frac{A}{B} \oplus \frac{C}{D} \right) = MS(f) \left(\frac{A}{B} \right) \oplus MS(f) \left(\frac{C}{D} \right)$$
$$\Leftrightarrow \frac{f(A \uplus C)}{f(B \uplus D)} = \frac{f(A)}{f(B)} \oplus \frac{f(C)}{f(D)}$$
$$\Leftrightarrow \frac{f(A \uplus C)}{f(B \uplus D)} = \frac{f(A) \uplus f(C)}{f(B) \uplus f(D)}$$

By Proposition 2.1.6, we have $f(A \uplus B) = f(A) \uplus f(B)$. This completes the proof that $MS(f)$ respects the direct sum.

3. $MS(f)$ must map tensor products to a tensor products. In other words,

$$MS(f) \left(\frac{A}{B} \otimes \frac{C}{D} \right) = MS(f) \left(\frac{A}{B} \right) \otimes MS(f) \left(\frac{C}{D} \right)$$
$$\Leftrightarrow MS(f) \left(\frac{A \bullet C \uplus B \bullet D}{A \bullet D \uplus B \bullet C} \right) = \frac{f(A)}{f(B)} \otimes \frac{f(C)}{f(D)}$$
$$\Leftrightarrow \frac{f(A \bullet C \uplus B \bullet D)}{f(A \bullet D \uplus B \bullet C)} = \frac{f(A) \bullet f(C) \uplus f(B) \bullet f(D)}{f(A) \bullet f(D) \uplus f(B) \bullet f(C)}$$

By Propositions 2.1.6 and 2.1.7, the left-hand side expands to equal the right-hand side.

4. Finally, $MS(f)$ must commute with the lambda-operations. Due to Theorem 2.1.2, and due to Lemma 2.2.7, it is sufficient for $MS(f)$ to commute with the Adams operations. That is,

$$\psi^n \left(MS(f) \left(\frac{A}{B} \right) \right) = MS(f) \left(\psi^n \left(\frac{A}{B} \right) \right)$$

An easy way to prove this is using multiplicity functions. Let $\psi^n(A)$, where A is a multiset, be the multiset obtained by raising every element of A to the power of n. Let's rewrite the right-hand side first using this:

$$MS(f) \left(\psi^n \left(\frac{A}{B} \right) \right) = MS(f) \left(\frac{\psi^n(A)}{\psi^n(B)} \right) = \frac{f(\psi^n(A))}{f(\psi^n(B))}$$

Now the left-hand side:

$$\psi^n \left(MS(f) \left(\frac{A}{B} \right) \right) = \psi^n \left(\frac{f(A)}{f(B)} \right) = \frac{\psi^n(f(A))}{\psi^n(f(B))}$$

Thus, it is enough to prove is that for all multisets A we have $f(\psi^n(A)) = \psi_n(f(A))$ for all positive integers n. This is clearly given by $f(a^n) = f(a)^n$ (which will be proven shortly), but can also use multiplicity functions to prove this. By definition 2.1.20, our statement is equivalent to

$$\sum_{\substack{a \in M \\ a^n = x}} 1_{f(A)}(a) = \sum_{\substack{a \in M \\ f(a) = x}} 1_{\psi^n(A)}(a)$$

which furthermore can be rewritten as

$$\sum_{\substack{a \in M \\ \psi^n(a) = x}} \sum_{\substack{b \in M \\ f(b) = a}} 1_A(b) = \sum_{\substack{a \in M \\ f(a) = x}} \sum_{\substack{b \in M \\ b^n = a}} 1_A(b)$$

$$\sum_{\substack{b \in M \\ f(b)^n = x}} 1_A(b) = \sum_{\substack{b \in M \\ f(b^n) = x}} 1_A(b)$$

Thus, it is sufficient to prove is $f(a)^n = f(a^n)$ for all a and $n \geq 1$. Let's simply use induction over n.

Base case: $n = 1$, $f(a^1) = f(a)^1$. This is trivial, since taking something to the power of 1 has no effect.

Inductive step: given that $f(a)^n = f(a^n)$, prove that $f(a)^{n+1} = f(a^{n+1})$. The latter can be rewritten as $f(a)^n \cdot f(a) = f(a^n \cdot a)$. The left hand side is again equal to $f(a^n) \cdot f(a)$. By the monoid homomorphism axioms, we have $f(a \cdot b) = f(a) \cdot f(b)$. This clearly proves our statement.

There are two more axioms we must prove for MS to be a functor.

1. $MS(id) = id$. In other words, for all motivic symbols A/B, we must have:

$$MS(id) \left(\frac{A}{B} \right) = \frac{A}{B}$$

$$\Leftrightarrow \frac{id(A)}{id(B)} = \frac{A}{B}$$

The latter follows from the definition of the identity function.

2. $MS(f \circ g) = MS(f) \circ MS(g)$. This can be proven by

$$MS(f \circ g)\left(\frac{A}{B}\right) = \frac{(f \circ g)(A)}{(f \circ g)(B)} = \frac{f(g(A))}{f(g(B))}$$

$$= MS(f)\left(MS(g)\left(\frac{A}{B}\right)\right) = (MS(f) \circ MS(g))\left(\frac{A}{B}\right)$$

With all of this, we have finally proven that MS is a functor. □

2.3 Properties of Motivic Symbols

In this section, we collect various minor results, observations and definitions on motivic symbols. None of these are used in the proof of the main theorem, but could perhaps have other applications.

2.3.1 Working with Motivic Pre-symbols

In this subsection, we outline basic lemmas surrounding equality of motivic pre-symbols in the context of normal motivic symbols.

For motivic symbols A/B and C/D,

$$A/B = C/D \Leftrightarrow A = C \wedge B = D$$

However, in many situations, we have cancellation of common elements before we get a final motivic symbol. Therefore, a statement like $A/B \otimes C/D = E/F$ cannot simply be stated as

$$E = A \bullet C \uplus B \bullet D \wedge F = A \bullet D \uplus B \bullet C$$

Instead, we can use the notion that such a motivic pre-symbol, for instance A/B, can also be written in the form $(C \uplus E)/(D \uplus E)$, where C/D is the actual motivic symbol, and E is the multiset that contains all common elements of C and D, or equivalently the intersection of C and D.

Lemma 2.3.1. Thus, if you have one motivic pre-symbol A/B equal to a motivic symbol C/D,

$$A/B = C/D \Leftrightarrow A = C \uplus E \wedge B = D \uplus E$$

$$\Rightarrow B \uplus C = A \uplus D$$

where $E = A \cap B$

Lemma 2.3.2. If you have two motivic pre-symbols A/B and C/D,

$$A/B = C/D \Leftrightarrow A \uplus F = C \uplus E \wedge B \uplus F = D \uplus E$$

$$\Leftrightarrow B \uplus C = A \uplus D$$

where $E = A \cap B$ and $F = C \cap D$

Lemma 2.3.3. If A/B is a motivic pre-symbol,

$$A/B = \varnothing/\varnothing \Leftrightarrow A = B$$

These are useful for proving results that rely on equalities involving operations that include cancellation of common elements.

2.3.2 Homomorphisms between MS-Rings

In this subsection, we prove a property of homomorphisms between MS-rings. We have so far not been able to provide a complete description of these homomorphisms. We do have some conjectures, but so far, this is the only proven theorem we have.

Theorem 2.3.4. For any monoids M and N, all lambda-ring homomorphisms $\varphi : MS(M) \to MS(N)$ will map $\{a\}/\varnothing$ to something with superdimension either $(n+1, n)$ or (n, n), for some natural number n.

Proof. Assume that $\varphi(\{a\}/\varnothing) = B/C$. By the axioms of lambda-ring homomorphisms, we know the following:

1. $\varphi(\{1_M\}/\varnothing) = \{1_N\}/\varnothing$
2. $\varphi(S_0 \oplus S_1) = \varphi(S_0) \oplus \varphi(S_1)$
3. $\varphi(S_0 \otimes S_1) = \varphi(S_0) \otimes \varphi(S_1)$
4. $\varphi(\psi^n(S)) = \psi^n(\varphi(S))$

By 3 and our assumption,

$$(\star) \quad \varphi\left(\frac{\{a\}}{\varnothing} \otimes \frac{\{a\}}{\varnothing}\right) = \frac{B}{C} \otimes \frac{B}{C} = \frac{B \bullet B \uplus C \bullet C}{B \bullet C \uplus B \bullet C}$$

Let $\psi^n(A)$, where A is a multiset, denote the multiset given by raising every element of A to the power of n. Note that the left-hand side in the equation above computes to $\varphi\left(\{a^2\}/\varnothing\right)$. This can be rewritten, using 4 and our assumption, as:

$$\varphi\left(\frac{\{a^2\}}{\varnothing}\right) = \varphi\left(\psi^2\left(\frac{\{a\}}{\varnothing}\right)\right) = \psi^2\left(\varphi\left(\frac{\{a\}}{\varnothing}\right)\right) = \psi^2\left(\frac{B}{C}\right) = \frac{\psi^2(B)}{\psi^2(C)}$$

By $(\star)$, we get

$$\frac{B \bullet B \uplus C \bullet C}{B \bullet C \uplus B \bullet C} = \frac{\psi^2(B)}{\psi^2(C)} \Leftrightarrow \frac{B \bullet B \uplus C \bullet C \uplus \psi^2(B)}{B \bullet C \uplus B \bullet C \uplus \psi^2(C)} = \frac{\varnothing}{\varnothing}$$

$$\Leftrightarrow B \bullet B \uplus C \bullet C \uplus \psi^2(C) = B \bullet C \uplus B \bullet C \uplus \psi^2(B)$$

From this, we can set up an equation of cardinality. Let $b := \#B$ and $c := \#C$. Then,

$$\Rightarrow b \cdot b + c \cdot c + c = b \cdot c + b \cdot c + b$$

$$\Leftrightarrow b^2 + c^2 + c = 2bc + b$$

$$\Leftrightarrow b - c = b^2 + c^2 - 2bc = (b - c)^2$$

$$\Leftrightarrow (b-c)^2 - (b-c) = (b-c)(b-c-1) = 0$$
$$\Leftrightarrow b-c = 0 \vee b-c-1 = 0$$
$$\Leftrightarrow b = c \vee b = c+1$$

This means that that the superdimension of B/C is either $(n+1, n)$ or (n, n), for some natural number n. □

2.3.3 Maps from MS-Rings to their underlying monoid

In this subsection we define three interesting maps from MS-Rings to their underlying structure. That is, a map $MS(M) \to M$. We also outline some properties of these maps. None of these are used later in this paper.

Definition 2.3.1. We define the absolute product map as the map

$$| - | : MS(M) \to M$$
$$\frac{A}{B} \mapsto \prod_{a \in A} a \cdot \prod_{b \in B} b$$

In other words, it maps a motivic symbol to the monoid product of all the elements in both the upstairs and downstairs. We denote this function of S as $|S|$.

Proposition 2.3.5.

1. Additive identity goes to multiplicative identity:
$$\left| \frac{\varnothing}{\varnothing} \right| = 1$$
2. Preservation of multiplicative identity:
$$\left| \frac{\{1\}}{\varnothing} \right| = 1$$
3. Evenness:
$$| \ominus S| = |S|$$
4. Individual powers to total power:
$$|\psi^n(S)| = |S|^n$$

Note 7. It is possible to describe $|A \oplus B|$ and $|A \otimes B|$ as well, but these expressions quickly become messy since cancellation is required when adding or multiplying A and B.

Proof.

1. This follows straight from the definition. The product of nothing is 1.

2. This also follows straight from the definition.

3. Using commutativity of $a \cdot b$:

$$\left|\ominus \frac{A}{B}\right| = \left|\frac{B}{A}\right| = \prod_{b\in B} b \cdot \prod_{a \in A} a = \prod_{a\in A} a \cdot \prod_{b\in B} b = \left|\frac{A}{B}\right|$$

4. Let $\psi^n(A)$, where A is a multiset, denote the multiset given by raising every element of A to the power of n. Using $a^n \cdot b^n = (a \cdot b)^n$:

$$\left|\psi^n\left(\frac{A}{B}\right)\right| = \left|\frac{\psi^n(A)}{\psi^n(B)}\right| = \prod_{a\in\psi^n(A)} a \cdot \prod_{b\in\psi^n(B)} b$$

$$= \prod_{a\in A} a^n \cdot \prod_{b\in B} b^n = \left(\prod_{a\in A} a \cdot \prod_{b\in B} b\right)^n = \left|\frac{A}{B}\right|^n$$

□

Definition 2.3.2. Given an abelian group G, we define the superdeterminant as:

$$\pi : MS(G) \to G$$
$$\frac{A}{B} \mapsto \prod_{a\in A} a \cdot \left(\prod_{b\in B} b\right)^{-1}$$

In other words, this is almost the same construction as the previous one, the difference being that we "divide" by the downstairs elements instead of multiplying by them.

Proposition 2.3.6.

1. Additive identity goes to multiplicative identity:

$$\left(\frac{\varnothing}{\varnothing}\right) = 1$$

2. Preservation of multiplicative identity:

$$\left(\frac{\{1\}}{\varnothing}\right) = 1$$

3. Additive inverse goes to multiplicative inverse:

$$\pi(\ominus S) = \pi(S)^{-1}$$

4. Individual powers to total power:

$$\pi(\psi^n(S)) = \pi(S)^n$$

5. Addition to multiplication:

$$\pi\left(\frac{A}{B} \oplus \frac{C}{D}\right) = \pi\left(\frac{A}{B}\right) \cdot \pi\left(\frac{C}{D}\right)$$

6. Multiplication to powered multiplication:

$$\pi\left(\frac{A}{B} \otimes \frac{C}{D}\right) = \pi\left(\frac{A}{B}\right)^{\#C-\#D} \cdot \pi\left(\frac{C}{D}\right)^{\#A-\#B}$$

Proof.

1. This follows straight from the definition. The product of nothing is 1.
2. This also follows straight from the definition.
3. Using commutativity of product in a monoid and $(a^{-1})^{-1} = a$:

$$\pi\left(\ominus\frac{A}{B}\right) = \pi\left(\frac{B}{A}\right) = \prod_{b\in B} b \cdot \left(\prod_{a\in A} a\right)^{-1} = \left(\prod_{a\in A} a \cdot \left(\prod_{b\in B} b\right)^{-1}\right)^{-1} = \pi\left(\frac{A}{B}\right)^{-1}$$

4. Let $\psi^n(A)$, where A is a multiset, denote the multiset given by raising every element of A to the power of n. Using $a^n \cdot b^n = (a \cdot b)^n$:

$$\pi\left(\psi^n\left(\frac{A}{B}\right)\right) = \pi\left(\frac{\psi^n(A)}{\psi^n(B)}\right) = \prod_{a\in\psi^n(A)} a \cdot \left(\prod_{b\in\psi^n(B)} b\right)^{-1}$$

$$= \prod_{a\in A} a^n \cdot \left(\prod_{b\in B} b^n\right)^{-1} = \left(\prod_{a\in A} a \cdot \left(\prod_{b\in B} b\right)^{-1}\right)^n = \pi\left(\frac{A}{B}\right)^n$$

5. Using associativity and commutativity of the product in a monoid:

$$\pi\left(\frac{A \uplus C}{B \uplus D}\right) = \prod_{a\in A\uplus C} a \cdot \left(\prod_{b\in B\uplus D} b\right)^{-1} = \prod_{a\in A} a \cdot \prod_{c\in C} c \cdot \left(\prod_{b\in B} b \cdot \prod_{d\in D} d\right)^{-1}$$

$$= \prod_{a\in A} a \cdot \left(\prod_{b\in B} b\right)^{-1} \cdot \prod_{c\in C} c \cdot \left(\prod_{d\in D} d\right)^{-1} = \pi\left(\frac{A}{B}\right) \cdot \pi\left(\frac{C}{D}\right)$$

6.

$$\pi\left(\frac{A}{B} \otimes \frac{C}{D}\right) = \pi\left(\frac{A \bullet C \uplus B \bullet D}{A \bullet D \uplus B \bullet C}\right) = \prod_{a\in(A\bullet C\uplus B\bullet D)} a \cdot \left(\prod_{b\in(A\bullet D\uplus B\bullet C)} b\right)^{-1} =$$

$$= \prod_{a\in A\bullet C} a \cdot \prod_{b\in B\bullet D} b \cdot \left(\prod_{c\in A\bullet D} c \cdot \prod_{d\in B\bullet C} d\right)^{-1}$$

In a single instance of $\prod_{a \in A \bullet B} a$, we have $\#A$ instances of the product of everything in B, and $\#B$ instances of the product of everything in A. In other words,

$$\prod_{a \in A \bullet B} a = \left(\prod_{a \in A} a\right)^{\#B} \cdot \left(\prod_{b \in B} b\right)^{\#A}$$

Applying this, we get:

$$= \left(\prod_{a \in A} a\right)^{\#C} \cdot \left(\prod_{c \in C} c\right)^{\#A} \cdot \left(\prod_{b \in B} b\right)^{\#D} \cdot \left(\prod_{d \in D} d\right)^{\#B}$$

$$\cdot \left(\left(\prod_{a \in A} a\right)^{\#D} \cdot \left(\prod_{d \in D} d\right)^{\#A} \cdot \left(\prod_{b \in B} b\right)^{\#C} \cdot \left(\prod_{c \in C} c\right)^{\#B}\right)^{-1}$$

$$= \left(\prod_{a \in A} a\right)^{\#C-\#D} \cdot \left(\prod_{c \in C} c\right)^{\#A-\#B} \cdot \left(\prod_{b \in B} b\right)^{\#D-\#C} \cdot \left(\prod_{d \in D} d\right)^{\#B-\#A}$$

$$= \left(\prod_{a \in A} a \cdot \left(\prod_{b \in B} b\right)^{-1}\right)^{\#C-\#D} \cdot \left(\prod_{c \in C} c \cdot \left(\prod_{d \in D} d\right)^{-1}\right)^{\#A-\#B}$$

$$= \pi\left(\frac{A}{B}\right)^{\#C-\#D} \cdot \pi\left(\frac{C}{D}\right)^{\#A-\#B}$$

□

Note 8. In the following definition, MS of a ring follows the convention laid out in Remark 1 of considering the multiplicative structure.

Definition 2.3.3. Given a ring R, we can construct the function

$$\begin{aligned} \varphi :& MS(R) \to R \\ \frac{A}{B} &\mapsto \sum_{a \in A} a - \sum_{b \in B} b \end{aligned}$$

In other words, this function maps a motivic symbol to the sum of all the elements upstairs, minus the sum of all elements downstairs. We call this function supertrace.

Proposition 2.3.7. This function is a ring homomorphism. That is:

1. $\varphi(\{1\}/\varnothing) = 1$
2. $\varphi(S_0 \oplus S_1) = \varphi(S_0) \oplus \varphi(S_1)$
3. $\varphi(S_0 \otimes S_1) = \varphi(S_0) \otimes \varphi(S_1)$

Proof.

1. This is given for free because the sum of 1 and nothing else is 1.

2. Using the ring axioms:

$$\varphi\left(\frac{A}{B}\oplus\frac{C}{D}\right)=\varphi\left(\frac{A\uplus C}{B\uplus D}\right)=\sum_{a\in A\uplus C}a-\left(\sum_{b\in B\uplus D}b\right)$$

$$=\sum_{a\in A}a+\sum_{c\in C}c-\left(\sum_{b\in B}b+\sum_{d\in D}d\right)=\sum_{a\in A}a-\left(\sum_{b\in B}b\right)+\sum_{c\in C}c-\left(\sum_{d\in D}d\right)$$

$$=\varphi\left(\frac{A}{B}\right)+\varphi\left(\frac{C}{D}\right)$$

3. Using the ring axioms:

$$\varphi\left(\frac{A}{B}\otimes\frac{C}{D}\right)=\varphi\left(\frac{A\bullet C\uplus B\bullet D}{A\bullet D\uplus B\bullet C}\right)=\varphi\left(\frac{A\bullet C}{A\bullet D}\right)+\varphi\left(\frac{B\bullet D}{B\bullet C}\right)$$

$$=\sum_{a\in A\bullet C}a-\left(\sum_{b\in A\bullet D}b\right)+\sum_{c\in B\bullet D}c-\left(\sum_{d\in B\bullet C}d\right)$$

A single instance of $\sum_{a\in A\bullet B}a$. $A\bullet B$, by definition, consists of all products that can be made using one element from A and one from B. Therefore, we can factor this as: $(\sum_{a\in A}a)\cdot(\sum_{b\in B}b)$ Let's apply this:

$$=\left(\sum_{a\in A}a\right)\cdot\left(\sum_{c\in C}c\right)-\left(\sum_{a\in A}a\right)\cdot\left(\sum_{d\in D}d\right)$$

$$+\left(\sum_{b\in B}b\right)\cdot\left(\sum_{d\in D}d\right)-\left(\sum_{b\in B}b\right)\cdot\left(\sum_{c\in C}c\right)$$

This can be factored as:

$$=\left(\sum_{a\in A}a-\sum_{b\in B}b\right)\cdot\left(\sum_{c\in C}c-\sum_{d\in D}d\right)$$

$$=\left(\sum_{a\in A}a-\sum_{b\in B}b\right)\cdot\left(\sum_{c\in C}c-\sum_{d\in D}d\right)=\varphi\left(\frac{A}{B}\right)\cdot\varphi\left(\frac{C}{D}\right)$$

□

2.3.4 Properties of the Functor

The aim of this subsection is to discuss some basic propositions about the functor MS. We also define the category $\widehat{MS}$, which is not directly used in this paper, but is something we plan to expand on in later publications.

Proposition 2.3.8. MS does not preserve limits. (see "Preservation of limits" in [8] for a definition on limit preservation).

Proof. Assume MS preserves limits. That implies $MS(X) \times MS(Y) \cong MS(X \times Y)$. This is not true. A simple counterexample is letting X and Y equal the monoid with one element. This implies that $MS(X) = MS(Y) \cong \mathbb{Z}$ (see Example 2.2.3). The product $X \times Y$ is of course isomorphic to both X and Y, which means that the above statement in this case is $\mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}$. This is most certainly not true. □

Definition 2.3.4. The category of MS-Rings, denoted by $\widehat{MS}$, is the subcategory of $Ring^{\lambda}$ which contains only the lambda-rings and lambda-ring homomorphisms that can be reached by the MS functor.

2.4 Corresponding Sequences

In this section, we define corresponding sequences and also construct a general approach to calculating them. We outline various properties, including theorems surrounding their uniqueness, and how various operations on motivic symbols affect their corresponding sequences.

Before starting, there are a few things that need to be cleared up. Although MS is defined on monoids, we extend the definition to commutative rings as well. This is done by simply forgetting the additive structure, and passing the multiplicative monoid through the functor (see Remark 1). Also, in the definition below we use the symbol t. This should not be thought of as an arbitrary element in R, but rather a separate formal symbol, with the intention of making formal power series work. Therefore, the definition below is really an equality in $R[[t]]$, with two elements equal if all coefficients are equal.

A final remark is that when talking about polynomials in this section, what is actually meant is a formal power series with coefficients of zero above a certain degree. We think of $R[t]$ as a subring of $R[[t]]$. When talking about factors or irreducibility of those polynomials, we refer to factors in $R[t]$.

2.4.1 Definition

In this subsection we define corresponding sequences, and prove that all corresponding sequences start with 1.

Definition 2.4.1. Let $\{c_n\}_0^\infty$ be a sequence of elements in a ring R. We say that the sequence $\{c_n\}$ corresponds to the motivic symbol $A/B \in MS(R)$ if (and only if)

$$G(c_n; t) \cdot \prod_{a \in A} (1 - a \cdot t) = \prod_{b \in B} (1 - b \cdot t)$$

Here, $G(c_n; t)$ refers to the formal power series with coefficients c_n for t^n, and 1 is the multiplicative identity in R.

Note 9. In this context, we are working with polynomials and formal power series. This means that equality is defined by equality of all coefficients, not functional equality.

Example 2.4.1. Let's look at a simple example. Let R equal $\mathbb{C}$, and $\{c_n\}$ be a sequence corresponding to the motivic symbol $\{z\}/\varnothing$. Using the definition:

$$G(c_n;t) \cdot \prod_{a\in\{z\}} (1 - a \cdot t) = \prod_{b\in\varnothing} (1 - b \cdot t)$$

$$\Leftrightarrow G(c_n;t) \cdot (1 - z \cdot t) = 1$$

We know that in $\mathbb{C}[[t]]$, $(1-z\cdot t)^{-1} = 1+zt+z^2t^2+z^3t^3\dots$ (here, $(\dots)^{-1}$ refers to multiplicative inverse). This implies that $1 = (1-z\cdot t)(1+zt+z^2t^2+z^3t^3+\dots)$. Thus, $G(c_n;t) = 1 + zt + z^2t^2 + z^3t^3 + \dots$, which means that $c_n = z^n$. This is a geometric sequence.

Lemma 2.4.1. The coefficient of t^0 in $\prod_{a\in A}(1 - a \cdot t)$ is 1, where A is a finite multiset.

Proof. This is trivial, given that the coefficient of t^0 is 1 for all the factors. For the sake of rigor however, one can prove it using a sort of induction on the cardinality of A. □

Proposition 2.4.2. If the sequence $\{c_n\}$ corresponds to a motivic symbol, then $c_0 = 1$.

Proof. Assume that $\{c_n\}$, corresponds to an arbitrary motivic symbol A/B. Then, by definition, and by Lemma 2.4.1, for some a_i and b_i,

$$(c_0 + c_1t + \dots) \cdot (1 + a_1t + a_2t^2 + \dots + a_kt^k) = (1 + b_1t + b_2t^2 + \dots + b_kt^k)$$

The coefficient of t^0 on the right-hand side, is of course 1. On the left-hand side, we have a product of a formal power series and a polynomial. To compute this one would use distribution, as given by the ring axioms. All the terms are of the form $a \cdot t^n$, for some element a in R, and some non-negative integer n. The product of two terms of the form $a \cdot t^n$ and $b \cdot t^m$, must be something of the form $a \cdot b \cdot t^{n+m}$. For the product to be of the form $a \cdot t^0$ (and thus affecting the coefficient of t^0), $n + m = 0$. Since n and m are non-negative, both have to be zero. This means that the coefficient of the left-hand side is built by the two terms c_0 and 1. Therefore, $c_0 \cdot 1 = 1$. 1 is the multiplicative identity, so $c_0 = 1$. □

2.4.2 General Construction and The Uniqueness Theorems

The aim of this subsection is to construct a general approach to finding a sequence corresponding to any motivic symbol, and to show that such a sequence is unique. Also, we show that if R is an integral domain, a sequence only corresponds to one motivic symbol, ignoring the ones that contain zeroes.

Let's try to construct a general approach to finding a sequence c_n corresponding to an arbitrary motivic symbol A/B. From the definition and from Proposition 2.4.2,

$$(1 + c_1t + c_2t^2 + c_3t^3 + \dots) \cdot \prod_{a\in A} (1 - a \cdot t) = \prod_{b\in B} (1 - b \cdot t)$$

Let's reformulate a few things. We know that both $\prod_{a\in A}(1-a\cdot t)$ and $\prod_{b\in B}(1-b\cdot t)$ are polynomials. Let's refer to these by the sequence of their coefficients, a_n and b_n respectively. Note that $a_0=b_0=1$. Then, the definition above can be rewritten as:

$$(1+c_1t+c_2t^2+c_3t^3+\ldots)\cdot(1+a_1t+a_2t^2+a_3t^3+\ldots+a_dt^d)=(1+b_1t+b_2t^2+b_3t^3+\ldots+b_et^e)$$

This can be split up into separate equations for every power of t. The degree of $(1+a_1t+a_2t^2+a_3t^3+\ldots+a_dt^d)$ is d. Then, we know by the distributive property that for natural numbers $n\geq d$:

$$(c_{n-d}\cdot a_d+c_{n-(d-1)}\cdot a_{d-1}+\ldots+c_n\cdot a_0)=b_n$$

Since $a_0=1$, we get

$$c_n=b_n-(c_{n-d}\cdot a_d+c_{n-(d-1)}\cdot a_{d-1}+\ldots+c_{n-1}\cdot a_1)$$

Thus we have a recursive definition of c_n for $n\geq d$, but how about when $n<d$? In these cases we can use the same definition, but replace c_j, where $j<0$ with 0, the additive identity in R. This is easily verifiable using the distributive property. For the sake of intuition, one can think the the coefficients of t^n for $n<0$ as 0, since they don't exist. From this, we have a general approach to constructing a corresponding sequence for an arbitrary motivic symbol in $MS(R)$, where R is a ring.

Theorem 2.4.3. There exists a corresponding sequence for all motivic symbols.

Proof. See construction above. □

Theorem 2.4.4. There is at most one sequence corresponding to a motivic symbol $A/B\in MS(R)$.

Proof. Assume there are two different sequences a_n and b_n, both corresponding to A/B. Then, by definition,

$$G(a_n;t)\cdot\prod_{a\in A}(1-a\cdot t)=\prod_{b\in B}(1-b\cdot t)$$

$$G(b_n;t)\cdot\prod_{a\in A}(1-a\cdot t)=\prod_{b\in B}(1-b\cdot t)$$

Note that the right-hand side is equal in both equations, so this implies

$$G(a_n;t)\cdot\prod_{a\in A}(1-a\cdot t)=G(b_n;t)\cdot\prod_{a\in A}(1-a\cdot t)$$

In $R[[t]]$, a formal power series is invertible if (and only if) its constant term is invertible in R (see [9], "Inverting series"). The constant term (coefficient of t^0) in the expansion of $\prod_{a\in A}(1-a\cdot t)$ is 1, the multiplicative identity of R, which is clearly invertible. Thus, we multiply both sides of the equation by the inverse of $\prod_{a\in A}(1-a\cdot t)$ and get $G(a_n;t)=G(b_n;t)$. This is by definition, equivalent to $a_n=b_n$. This is a contradiction. □

Thus, we have proven that all motivic symbols in any ring correspond to exactly one sequence, which can be constructed as described above.

Definition 2.4.2. We denote the sequence corresponding to S by $Corr(S)$.

Lemma 2.4.5. Let R be a ring and $A/B, C/D \in MS(R)$. If the following holds

$$\prod_{x\in A\uplus D}(1-x\cdot t)=\prod_{y\in B\uplus C}(1-y\cdot t)\Rightarrow A\uplus D=B\uplus C \tag{I}$$

in the ring R, then

$$Corr\left(\frac{A}{B}\right)=Corr\left(\frac{C}{D}\right)\Rightarrow\frac{A}{B}=\frac{C}{D}$$

In other words, (I) implies $Corr$ is injective.

Proof. Assume $Corr(A/B) = Corr(C/D) =: \{c_n\}$. Then, by definition

$$G(c_n;t)\cdot\prod_{a\in A}(1-a\cdot t)=\prod_{b\in B}(1-b\cdot t)$$

$$G(c_n;t)\cdot\prod_{c\in C}(1-c\cdot t)=\prod_{d\in D}(1-d\cdot t)$$

By invertibility due to the constant term being 1,

$$G(c_n;t)=\prod_{b\in B}(1-b\cdot t)\cdot\prod_{a\in A}(1-a\cdot t)^{-1}$$

$$G(c_n;t)=\prod_{d\in D}(1-d\cdot t)\cdot\prod_{c\in C}(1-c\cdot t)^{-1}$$

This implies

$$\prod_{d\in D}(1-d\cdot t)\cdot\prod_{c\in C}(1-c\cdot t)^{-1}=\prod_{b\in B}(1-b\cdot t)\cdot\prod_{a\in A}(1-a\cdot t)^{-1}$$

$$\Leftrightarrow\prod_{d\in D}(1-d\cdot t)\cdot\prod_{a\in A}(1-a\cdot t)=\prod_{b\in B}(1-b\cdot t)\cdot\prod_{c\in C}(1-c\cdot t)$$

$$\Leftrightarrow\prod_{a\in A\uplus D}(1-a\cdot t)=\prod_{b\in B\uplus C}(1-b\cdot t)$$

Because of our axiom, this implies $A \uplus D = B \uplus C$. Since A and B are disjoint, $D = B \uplus E$, where E is some submultiset of C. Since C and D are disjoint, $E = \varnothing$. This also gives $A = C$. Combining those, we get $A/B = C/D$. □

Recall that we define the elementary symmetric polynomials e_n in Definition 2.1.7.

Lemma 2.4.6. Let U, V be multisets consisting of elements from a ring.

$$\prod_{u\in U}(1-u\cdot t)=\prod_{v\in V}(1-v\cdot t)$$

if and only if

$$e_n(U)=e_n(V)\qquad\forall n$$

Proof. Assume

$$\prod_{u \in U} (1 - u \cdot t) = \prod_{v \in V} (1 - v \cdot t)$$

Let's use elementary algebra to expand the products. According to the "Properties" section in [6], we have the following:

$$\prod_{j=1}^{n} (\lambda - X_j) = \lambda^n - e_1(X_1, \ldots, X_n)\lambda^{n-1} + \ldots + (-1)^n e_n(X_1, \ldots, X_n)$$

Here, $e_n(\ldots)$ refers to the n-th elementary symmetric polynomials of the variables in the parentheses. We apply this to the left-hand side of our statement, with $\lambda = 1$, $n = \#U$, and $X_j = u_j \cdot t$:

$$\prod_{u \in U} (1 - u \cdot t) =$$

$$1 - e_1(u_1 \cdot t, \ldots, u_n \cdot t) + e_2(u_1 \cdot t, \ldots, u_n \cdot t) - \ldots + (-1)^n e_n(u_1 \cdot t, \ldots, u_n \cdot t)$$
$$= 1 - e_1(u_1, \ldots, u_n)t + e_2(u_1, \ldots, u_n)t^2 - \ldots + (-1)^n e_n(u_1, \ldots, u_n)t^n$$

$$= 1 - e_1(U)t + e_2(U)t^2 - \ldots + (-1)^{\#U} e_{\#U}(U)t^{\#U}$$

where we by $e_n(U)$ denote $e_n(u_1, u_2, \ldots, u_{\#U})$ where $u_1, u_2, \ldots, u_{\#U}$ are all the elements of the multiset U in an arbitrary order. We can easily rewrite the right-hand side with a more or less identical technique, so we are left with:

$$1 - e_1(U)t + e_2(U)t^2 - \ldots + (-1)^{\#U} e_{\#U}(U)t^{\#U} =$$

$$1 - e_1(V)t + e_2(V)t^2 - \ldots + (-1)^{\#V} e_{\#V}(V)t^{\#V}$$

Since we are dealing with polynomial equality, we get the following equations for all $n \geq 0$:

$$e_n(U) = e_n(V)$$

where we define $e_n(U) = 0$ for $n > \#U$ and $e_0(U) = 1$.

We also need to prove the implication in the other direction. Assume $e_n(U) = e_n(V)\ \forall n \geq 0$. By the fundamental theorem of symmetric polynomials (see [6], section "The fundamental theorem of symmetric polynomials"), we can write all symmetric polynomials using the elementary ones. □

Theorem 2.4.7. Let R be an integral domain, and $A/B, C/D \in MS(R)$ but without any instances of the element 0 upstairs or downstairs. Then,

$$Corr\left(\frac{A}{B}\right) = Corr\left(\frac{C}{D}\right) \Rightarrow \frac{A}{B} = \frac{C}{D}$$

In other words, for integral domains, *Corr* considered as function is injective. Equivalently, all sequences correspond to at most one motivic symbol.

Proof. Let R be an integral domain, and $A/B, C/D \in MS(R)$. By Lemma 2.4.5, it is sufficient to prove that

$$\prod_{x\in A\uplus D}(1-x\cdot t)=\prod_{y\in B\uplus C}(1-y\cdot t)\Rightarrow A\uplus D=B\uplus C$$

By Lemma 2.4.6, we can simplify this to proving that $\forall n:\ e_n(A\uplus D)=e_n(B\uplus C)\Rightarrow A\uplus D=B\uplus C$. This is given by proving that for all multisets U and V without zeroes, $e_n(U)=e_n(V)\ \forall n\Rightarrow U=V$. To prove this implication, assume

$$e_n(U)=e_n(V)\qquad \forall n$$

First of all, we will prove that this implies equal cardinality of U and V. Recall that we do not allow zeroes in these multisets. Assume U and V have different cardinality, namely a and b respectively. Assume $a>b$. If $b>a$, simply swap U and V. Note that by our assumption $a\neq b$. Now, $e_a(A)$ is the product of all elements in U, since a is the cardinality of U. On the other hand, $e_a(V)=0$ because $a>\#V$. Since $e_n(U)=e_n(V)$ for all n, including a, we have the product of all elements in U equal to 0. In an integral domain, this implies that at least one element of U is equal to 0, which is a contradiction since U by definition does not have any zeroes. Therefore, we have $\#U=\#V$

Given an element $x\in R$, construct the product

$$\prod_{u\in U}(x-u)$$

Note that this is a symmetric polynomial over the values of U, so the coefficient of each x^k is built entirely of elementary symmetric polynomials of the elements of U (see [6], "The fundamental theorem of symmetric polynomials"). Because U and V have the same cardinality, we can construct the same product with elements of V in the exact same manner. Thus, we have

$$\prod_{u\in U}(x-u)=\prod_{v\in V}(x-v)$$

If we put $x=u$ for some $u\in U$, we get

$$\prod_{v\in V}(u-v)=0$$

In an integral domain, this implies that at least one term $(u-v)=0$ for some $v\in V$. Thus, for every element $u\in U$ we have an element $v\in V$ such that $u=v$. Removing these elements from U and V respectively, the cardinality of U drops by 1, and the same happens to V. Repeating this process, we end up with $\varnothing=\varnothing$, since U and V have the same cardinality to start with. Hence, the original multisets U and V are equal. □

Example 2.4.2. If R in Theorem 2.4.7 is not an integral domain, the above theorem might fail. For instance, in $\mathbb{Z}/4$, $e_n(1,1)=e_n(3,3)$ for all n, and thus $\{1,1\}/\varnothing$ and $\{3,3\}/\varnothing$ correspond to the same sequence in $MS(\mathbb{Z}/4)$ (by Lemmas 2.4.5 and 2.4.6).

So, to summarize:

1. Every motivic symbol corresponds to a sequence.
2. This sequence is always unique. In other words, there is exactly one sequence corresponding to a given motivic symbol.
3. In MS of an integral domain, sequences correspond to unique motivic symbols, given that we ignore motivic symbols with 0 upstairs or downstairs. In other words, two different motivic symbols can not correspond to the same sequence in an integral domain, given that neither contain 0 in the upstairs part or in the downstairs part.

2.4.3 Relation with Direct Sum and Tensor Product

In this subsection we explore how the tensor product and direct sum affect corresponding sequences.

Theorem 2.4.8. Let $S_0, S_1 \in MS(R)$, for some ring R. Then

$$Corr(S_0 \oplus S_1) = Corr(S_0) \cdot Corr(S_1)$$

In other words, the direct sum corresponds to Cauchy product of formal power series.

Proof. Let $\{c_n\} := Corr(A/B \oplus C/D)$ Let's use the definition of corresponding sequences to get:

$$G(c_n;t) \cdot \prod_{a \in A \uplus C} (1 - a \cdot t) = \prod_{b \in B \uplus D} (1 - b \cdot t)$$

$$= G(c_n;t) \cdot \prod_{a \in A} (1 - a \cdot t) \cdot \prod_{c \in C} (1 - c \cdot t) = \prod_{b \in B} (1 - b \cdot t) \cdot \prod_{d \in D} (1 - d \cdot t)$$

Let $\{a_n\} := Corr(A/B)$ and $\{b_n\} := Corr(C/D)$. Then, by definition, we have

$$G(a_n;t) \cdot \prod_{a \in A} (1 - a \cdot t) = \prod_{b \in B} (1 - b \cdot t)$$

$$G(b_n;t) \cdot \prod_{c \in C} (1 - c \cdot t) = \prod_{d \in D} (1 - d \cdot t)$$

Multiplying them together, we get

$$G(a_n;t) \cdot G(b_n;t) \cdot \prod_{a \in A} (1 - a \cdot t) \cdot \prod_{c \in C} (1 - c \cdot t) = \prod_{b \in B} (1 - b \cdot t) \cdot \prod_{d \in D} (1 - d \cdot t)$$

Note that all the factors on both the right and left-hand side have constant term 1 (as proven earlier), so all factors are invertible. Divide the first equation by this one, and cancel as much as possible. This leaves:

$$\frac{G(c_n;t)}{G(a_n;t) \cdot G(b_n;t)} = 1$$

$$\Leftrightarrow G(c_n;t) = G(a_n;t) \cdot G(b_n;t)$$

which is what we wanted to show. □

Note 10. This also gives us an expression for the sequence corresponding to an additive inverse. We know that $A/B \oplus \ominus A/B = \varnothing/\varnothing$. This gives us $Corr(A/B) \cdot Corr(\ominus A/B) = Corr(\varnothing/\varnothing)$. $Corr(\varnothing/\varnothing)$ is given by $G(c_n;t) \cdot 1 = 1$, so $Corr(\varnothing/\varnothing) = 1, 0, 0, 0, \ldots$. We are thus left with $G(Corr(A/B);t) \cdot G(Corr(\ominus A/B);t) = 1$, which means that $Corr(\ominus A/B)$ considered as a formal power series is the multiplicative inverse of $Corr(A/B)$.

Theorem 2.4.9. For a motivic symbol A/B and and an element x, the following holds for all indices $n \geq 0$:

$$Corr\left(\frac{\{x\}}{\varnothing} \otimes \frac{A}{B}\right)_n = Corr\left(\frac{A}{B}\right)_n \cdot x^n$$

Proof. Let

$$\{c_n\} := Corr\left(\frac{\{x\}}{\varnothing} \otimes \frac{A}{B}\right) = Corr\left(\frac{\{x\} \bullet A}{\{x\} \bullet B}\right)$$

By definition, we have:

$$G(c_n;t) \cdot \prod_{a \in \{x\} \bullet A} (1 - a \cdot t) = \prod_{b \in \{x\} \bullet B} (1 - b \cdot t)$$

$$\Leftrightarrow (c_n;t) \cdot \prod_{a \in A} (1 - a \cdot x \cdot t) = \prod_{b \in B} (1 - b \cdot x \cdot t)$$

Let $\{d_n\} := Corr(A/B)$. Recall that by definition,

$$(d_n;t) \cdot \prod_{a \in A} (1 - a \cdot t) = \prod_{b \in B} (1 - b \cdot t)$$

Note that x is constant, and a factor of t. This means that we can substitute t with $t \cdot x$, and receive an evaluation of $G(c_n;t)$ akin to that of $G(d_n;t)$, the only difference being that t in $G(c_n;t)$ has a factor x. In other words, $c_0 + c_1 t + c_2 t^2 + \ldots =: G(c_n;t) = d_0 + d_1(tx) + d_2(tx)^2 + \ldots = d_0 + (d_1 x)t + (d_2 x^2)t^2 + \ldots$. Thus, we simply have $c_n = d_n \cdot x^n$. □

Note 11. Terms of the form $\{x\}/\varnothing$ provide a basis for all motivic symbols. Specifically, all motivic symbols A/B can be written as:

$$\frac{A}{B} = \bigoplus_{a \in A} \left(\frac{\{a\}}{\varnothing}\right) \ominus \bigoplus_{b \in B} \left(\frac{\{b\}}{\varnothing}\right)$$

Since the tensor product distributes over direct sum, and since we have already described $Corr(S_0 \oplus S_1)$, this gives a full definition of the corresponding sequence of any tensor product.

Corollary 1. Let $A/B, C/D \in MS(R)$ for a ring R.

$$Corr\left(\frac{A}{B} \otimes \frac{C}{D}\right) = \prod_{a \in A} Corr\left(\frac{\{a\}}{\varnothing} \otimes \frac{C}{D}\right) \cdot \left(\prod_{b \in B} Corr\left(\frac{\{b\}}{\varnothing} \otimes \frac{C}{D}\right)\right)^{-1}$$

Here, the multiplicative inverse of a sequence is defined as its Cauchy product inverse.

2.4.4 The Ring Homomorphism Mapping Theorem

The aim of this subsection is to describe $Corr(MS(f)(S))$ for ring homomorphisms f. In other words, we describe what happens to the corresponding sequence when you map all the elements in a motivic symbol through a ring homomorphism.

Theorem 2.4.10. Let $S \in MS(R)$, where R is a ring, and let f be a ring homomorphism from R. Then,

$$Corr\left(MS(f)\left(S\right)\right)_n = f(Corr(S)_n)$$

In other words, sending the elements of a motivic symbol through a ring homomorphism is equivalent to sending every element of its corresponding sequence through the same ring homomorphism.

Proof. Let $S = A/B$. By definition,

$$MS(f)\left(\frac{A}{B}\right) = \frac{f(A)}{f(B)}$$

Also, let $Corr\left(MS(f)\left(\frac{A}{B}\right)\right) =: \{c_n\}$. Then we have:

$$G(c_n;t) \cdot \prod_{a \in f(A)} (1 - a \cdot t) = \prod_{b \in f(B)} (1 - b \cdot t)$$

$$\Leftrightarrow G(c_n;t) \cdot \prod_{a \in A} (1 - f(a) \cdot t) = \prod_{b \in B} (1 - f(b) \cdot t)$$

Given a ring homomorphism $f : R \to S$, we have a canonical ring homomorphism $f_t : R[[t]] \to S[[t]]$, with $f_t(x) = f(x)$ for $x \in R$ and $f_t(t) = t$. Using this and the ring homomorphism axioms:

$$G(c_n;t) \cdot \prod_{a \in A} (1 - f(a) \cdot t) = \prod_{b \in B} (1 - f(b) \cdot t)$$

$$\Leftrightarrow G(c_n;t) \cdot \prod_{a \in A} (1 - f_t(a) \cdot t) = \prod_{b \in B} (1 - f_t(b) \cdot t)$$

$$\Leftrightarrow G(c_n;t) \cdot \prod_{a \in A} (f_t(1) - f_t(a) \cdot f_t(t)) = \prod_{b \in B} (f_t(1) - f_t(b) \cdot f_t(t))$$

$$\Leftrightarrow G(c_n;t) \cdot \prod_{a \in A} (f_t(1) - f_t(a \cdot t)) = \prod_{b \in B} (f_t(1) - f_t(b \cdot t))$$

$$\Leftrightarrow G(c_n;t) \cdot \prod_{a \in A} f_t(1 - a \cdot t) = \prod_{b \in B} f_t(1 - b \cdot t)$$

$$\Longrightarrow G(c_n;t) \cdot f_t\left(\prod_{a \in A} (1 - a \cdot t)\right) = f_t\left(\prod_{b \in B} (1 - b \cdot t)\right)$$

Let $\{d_n\} := Corr(A/B)$. Then we have:

$$G(g_n;t) \cdot \prod_{a \in A} (1 - a \cdot t) = \prod_{b \in B} (1 - b \cdot t)$$

Which implies, by method of insertion,

$$G(c_n;t) \cdot f_t\left(\prod_{a\in A}(1-a\cdot t)\right) = f_t\left(G(d_n;t)\cdot\prod_{a\in A}(1-a\cdot t)\right)$$

Now, by the ring homomorphism axioms we have:

$$G(c_n;t) \cdot f_t\left(\prod_{a\in A}(1-a\cdot t)\right) = f_t\left(G(d_n;t)\right)\cdot f_t\left(\prod_{a\in A}(1-a\cdot t)\right)$$

Note that $f_t\left(\prod_{a\in A}(1-a\cdot t)\right)$ has an inverse due to having constant term 1. This is because f_t maps 1 to 1, and because it maps t^n to t^n. Thus,

$$G(c_n;t) = f_t\left(G(d_n;t)\right)$$

$$\Leftrightarrow 1 + c_1t + c_2t^2 + c_3t^3 \ldots = f_t(1 + d_1t + d_2t^2 + d_3t^3 \ldots)$$

Finally, by the ring homomorphism axioms:

$$1 + c_1t + c_2t^2 + c_3t^3 \ldots = f_t(1 + d_1t + d_2t^2 + d_3t^3 \ldots)\ldots$$

$$\Leftrightarrow 1 + c_1t + c_2t^2 + c_3t^3 \ldots = f_t(1) + f_t(d_1t) + f_t(d_2t^2) + f_t(d_3t^3)\ldots$$

$$\Leftrightarrow 1 + c_1t + c_2t^2 + c_3t^3 \ldots = f_t(1) + f_t(d_1)f_t(t) + f_t(d_2)f_t(t^2) + f_t(d_3)f_t(t^3)\ldots$$

$$\Leftrightarrow 1 + c_1t + c_2t^2 + c_3t^3 \ldots = 1 + f(d_1)t + f(d_2)t^2 + f(d_3)t^3\ldots$$

This implies $c_n = f(d_n)$ for all indices $n \geq 1$. □

Example 2.4.3. Important examples of ring homomorphisms are for instance complex conjugation, the canonical surjection $\mathbb{Z} \to \mathbb{Z}/n$, furthermore the unique homomorphism $\mathbb{Z} \to R$ for any ring R, and so on.

2.4.5 Extending The Definition

In this subsection we extend the definition of corresponding sequences to allow corresponding sequences in the context of monoids with functions into rings. We also reformulate all previous theorems in this new context.

Definition 2.4.3. Given a monoid $(M,\cdot)$, a ring $(R,+,\cdot)$, and a function f from M to R, as well as a motivic symbol $A/B \in MS(M)$, and a sequence $\{c_n\}$ with elements in R, we say that $\{c_n\}$ corresponds to A/B if (and only if)

$$G(c_n;t)\cdot\prod_{a\in A}(1-f(a)\cdot t) = \prod_{b\in B}(1-f(b)\cdot t)$$

Here, $G(c_n;t)$ refers to the formal power series with coefficients c_n for t^n, and 1 is the multiplicative identity in R. Simply put, we send the elements of the motivic symbol through f and compute corresponding sequences in R.

Remark 2. If f is a monoid homomorphism from $(M,\cdot)$ to $(R,\cdot)$ (the multiplicative structure of the ring R), one can refer the triple (M,R,f) as an affine pre-log scheme. It is possible that the functor MS can be refined to a functor from the category of affine pre-log schemes to the category of lambda-rings.

Note 12. All following theorems and propositions use the naming conventions from Definition 2.4.3.

Theorem 2.4.11. Proposition 2.4.2 and Theorems 2.4.3 and 2.4.4 hold in this context.

Proof. A corresponding sequence in $MS(M)$ is just a corresponding sequence in $MS(R)$ with the motivic symbol's elements coming from a function. Those statements concern any motivic symbol. Therefore, it is sufficient to simply see the proofs of those individual propositions and theorems. □

Definition 2.4.4. We define $Corr_f(A/B)$ to be the sequence corresponding to A/B. If the function f is clear from context, one can also write $Corr(A/B)$.

Theorem 2.4.12. If f is injective and R is an integral domain, sequences correspond to at most one motivic symbol in $MS(M)$, given that one ignores motivic symbols with elements that are mapped to 0.

Proof. This is similar to Proposition 2.4.11. The statement holds in $MS(R)$ due to Theorem 2.4.7 holding, and the only restriction on motivic symbols in the statement in $MS(R)$ is that they cannot contain zeroes. This restriction is clearly covered by ignoring motivic symbols that contain something mapped to 0. Therefore, one can simply see the proof of Theorem 2.4.7. □

Theorem 2.4.13. If R is an integral domain and f is injective and doesn't map any elements to 0, sequences correspond to at most one motivic symbol in $MS(M)$.

Proof. See Theorem 2.4.12. This is a special case of that. □

Theorem 2.4.14. All theorems and the corollary in Subsection 2.4.3 work in this context if f is a monoid homomorphism.

Proof. Note that due to Theorem 2.2.8, we have a lambda-ring homomorphism $MS(f) : MS(M) \to MS(R)$. This homomorphism is defined as applying f to all elements of the motivic symbol. This is exactly what we doing when defining corresponding sequences in $MS(M)$. In other words, we have

$$Corr(MS(f)(S)) = Corr_f(S)$$

If we let S be a direct sum or tensor product, the ring homomorphism axioms will allow you to split $MS(f)(S)$ into a direct sum or tensor product respectively. Then one can use the theorems, and finally apply $Corr(MS(f)(S)) = Corr_f(S)$ in order to state the theorem in this context. For the sake of an example, let's show this process for Theorem 2.4.8.

$$\begin{aligned} Corr_f(S_0 \oplus S_1) &= Corr(MS(f)(S_0 \oplus S_1)) = Corr(MS(f)(S_0) \oplus MS(f)(S_1)) \\ &= Corr(MS(f)(S_0)) \cdot Corr(MS(f)(S_1)) = Corr_f(S_0) \cdot Corr_f(S_1) \end{aligned}$$

□

Note 13. Theorem 2.4.10 is not possible to in a meaningful way bring into this context, because the motivation behind that theorem is describing what happens when you map elements of a motivic symbol through a ring homomorphism, which is undefined when M is a monoid.

Note 14. If M is a submonoid of the multiplicative structure of an integral domain without the additive identity 0, all of these theorems hold.

Chapter 3

Classical Multiplicative Functions

3.1 Multiplicative Functions and their Generating series

In this section we define the concept of a multiplicative function and certain generating series associated to such a function. When no other reference is given, the definitions are taken from Wikipedia and the book “Introduction to Arithmetical Functions” by Paul J. McCarthy [2].

3.1.1 Definitions

The aim of this section is to give some key definitions to be used in the discussion of multiplicative functions.

Definition 3.1.1. An arithmetical function is, as defined earlier, any function that takes a positive integer as input, and gives a complex number as output. In other words, it is a complex-valued function defined on the set of positive integers.

Definition 3.1.2. An arithmetical function f is said to be multiplicative if

1. $f(1) = 1$, and if
2. $f(mn) = f(m)f(n)$ for all m and n with $gcd(m, n) = 1$.

Definition 3.1.3. An multiplicative function f is called completely multiplicative if $f(mn) = f(m)f(n)$ for all m and n.

Example 3.1.1. The Liouville function is denoted by $\lambda(n)$. If n is a positive integer, then $\lambda(n)$ is defined as:

$$\lambda(n) = (-1)^{\Omega(n)},$$

where $\Omega(n)$ is the number of prime factors of n, counted with multiplicity. The Liouville function is completely multiplicative because $\Omega(mn) = \Omega(m) + \Omega(n)$ whether $gcd(m, n) = 1$ or not.

Definition 3.1.4. The delta function $\delta(n)$ is defined by

$$\begin{cases} 1 & \text{for } \delta(1) \\ 0 & \text{for } n \geq 2 \end{cases}$$

$$\delta(1) = 1, \delta(n) = 0 \text{ for } n \geq 2$$

Definition 3.1.5. If f and g are two arithmetical functions, one defines a new arithmetical function $f * g$, the Dirichlet convolution of f and g, by

$$\begin{aligned} (f * g)(n) &= \sum_{d \,|\, n} f(d) g\left(\frac{n}{d}\right) \\ &= \sum_{ab \,=\, n} f(a) g(b) \end{aligned}$$

where the sum extends over all positive divisors d of n, or equivalently over all distinct pairs (a, b) of positive integers whose product is n. If f and g are multiplicative, then so is $f * g$ (See [10], section "Properties").

It is well-known that the set of multiplicative functions is an abelian group under Dirichlet convolution, in which the delta function $\delta(n)$ (see Definition 3.1.4) is the identity element.

Definition 3.1.6. An arithmetical function g is the Dirichlet inverse of an arithmetical function f if $f * g = \delta$.

Proposition 3.1.1. An arithmetical function f has a Dirichlet inverse if and only id $f(1) \neq 0$.

Proof. See [2], proposition 1.2 □

Corollary 2. All multiplicative functions have Dirichlet inverses.

Proposition 3.1.2. If f is multiplicative, the the Dirichlet inverse of f will also be multiplicative.

Proof. See [2], proposition 1.5 □

Definition 3.1.7. The norm of a multiplicative function f is the function $N(f)$ defined by

$$N(f)(n) = \sum_{d|n^2} f(n^2/d)\lambda(d)f(d)$$

where $\lambda(d)$ is the Liouville function. Here the sum is taken over all positive divisors of n^2.

Proposition 3.1.3. The norm of a multiplicative function is a multiplicative function.

Proof. See [2], exercise 1.70. □

Note 15. The norm operator was studied by Sivaramakrishnan and Redmond in their paper "Some Properties of Specially Multiplicative Functions" [3], in which they also defined "higher norm" N^r by the recursive formula $N^r(f) = N(N^{r-1}(f))$. The relevance for us is that these higher norms correspond to the Adams operations ψ^2, ψ^4, ψ^8 and so on. More on this later.

3.1.2 Generating series

In this section we define two kinds of generating functions associated to a multiplicative function, called the Bell series and the Dirichlet series. We then show how these can be used to compute motivic symbols. The goal is to go from the multiplicative function to motivic symbols through computation of their Bell series or Dirichlet series.

Definition 3.1.8. Let f be a multiplicative function and p a prime. We define the formal power series, $f_p(t)$, called the Bell series of f at the prime p as

$$f_p(t) \to \sum_{n=0}^{\infty} f(p^n)t^n$$

Definition 3.1.9. A function f is said to have a rational Bell series for each prime p if the Bell series can be written as a rational expression. It is known that this is the equivalent to the sequence $\{f(p^n)\}$ being linearly recursive.

Example 3.1.2. The constant function $1(n)$ is defined by $1(n) = 1$ for all n. Its Bell coefficients are $1, 1, 1, \cdots$, giving us the Bell series $1 + t + t^2 + t^3 + \cdots$. This can also be written as $\frac{1}{1-t}$,therefore the function has a rational Bell series.

Example 3.1.3. The Möbius function $\mu : \mathbb{Z}_+ \to \mathbb{C}$ is defined by

- $\mu(n) = 1$ if n is a square-free positive integer with an even number of prime factors.
- $\mu(n) = -1$ if n is a square-free positive integer with an odd number of prime factors.
- $\mu(n) = 0$ if n has a squared prime factor.

When $n = p^k$, for prime number p, we get $\mu(1) = 1$, $\mu(p) = -1$, $\mu(p^k) = 0$ for $k \geq 2$.

Therefore the Bell series of the Möbius function is $\mu_p = 1 - t$. We know that the function is rational because it can also be written as $\frac{1-t}{1}$. Here the Bell series is independent of p.

Note 16. For many multiplicative functions, you can compute their corresponding Bell series without much effort, by finding the function values $f(p^e)$ and manipulating geometric series (see example in Table 3.1).

Definition 3.1.10. Let f be an arithmetical function. We define the Dirichlet series $D_f(s)$ of f to be

$$D_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

Remark 3. If f is multiplicative, this can be rewritten as an Euler product. More specifically:

$$D_f(s) = \prod_{p \text{ prime}} f_p\left(p^{-s}\right)$$

where f_p is the Bell series at p.

Remark 4. Although it is possible to view a Dirichlet series as a function of a complex variable s, we prefer to think of it as a formal series.

Definition 3.1.11. The Dirichlet series of the constant function will be

$$D_f(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

We define $\zeta(s)$ to be this Dirichlet series (when viewed as a function of a complex variable, this function is called the Riemann zeta function).

Example 3.1.4. The Dirichlet series of the Möbius function is

$$D_f(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

It can be written as the Euler product

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots$$

Example 3.1.5. The Euler totient function denoted by φ counts the positive integers less than or equal to n that are relatively prime to n. For $n = p^e$, there will be exactly $p^e - p^{e-1}$ integers that satisfies this demand, so we will get

$$\varphi(p^e) = p^e - \frac{p^e}{p}$$

The Bell series for the Euler totient function at $p = 2$ will be $f_2(t) = 1 + t + 2t^2 + 4t^3 + ...$ (See Table 3.1). This can be written as the rational Bell series

$$f_2(t) = \frac{1-t}{1-2t}$$

.

Table 3.1: Euler totient function at $p = 2$

	p^e	$f(p^e)$	$f(p^e)t^e$
e=0	1	1	1
e=1	2	1	t
e=2	4	2	$2t^2$
e=3	8	4	$4t^3$
e=4	16	8	$8t^3$
e=5	32	16	$16t^3$

Example 3.1.6. As we have already seen, the Bell series for the Möbius function at $p = 2$ will be $f_2(t) = 1 - t$ (see Table 3.2), which is obviously rational.

Table 3.2: Möbius function at $p = 2$

	p^e	$f(p^e)$	$f(p^e)t^e$
e=0	1	1	1
e=1	2	-1	$-t$
e=2	4	0	0
e=3	8	0	0

3.2 Motivic symbols from Bell series

In this section we explain what we mean by the motivic symbol of a Bell series and provide some simple examples.

We write $\mathbb{C}^*$ for the group of nonzero complex numbers.

Theorem 3.2.1. If $h(t)$ is a rational expression in t with complex coefficients, satisfying $h(0) = 1$, then there exists a unique motivic symbol such that

$$h(t) = G(Corr(A/B); t)$$

Proof. Such a rational expression can be written $u(t)/v(t)$ where u and v are polynomials with constant term 1. Over $\mathbb{C}$, all such polynomials can be uniquely factored into a product of the form $(1-a_1t)(1-a_2t)\dots$ with coefficients $a_i \in \mathbb{C}^*$. Let A be the multiset of coefficients from the denominator, and B be these coefficients from the numerator. Then the motivic symbol A/B clearly satisfies the condition in the theorem.

By Theorem 2.4.7, there is at most one motivic symbol satisfying this. □

Definition 3.2.1. Let f be a multiplicative function with rational Bell series and let p be a prime number. We define the motivic symbol of f at p as the motivic symbol $A/B \in MS(\mathbb{C}^*)$ satisfying

$$G(Corr(A/B); t) = f_p(t)$$

Example 3.2.1. The constant function $1(n)$ has the Bell series

$$1 + t + t^2 + \cdots + t^n = \frac{1}{1-t}$$

We have $u(t) = 1$ and $v(t) = 1 - t$. From $v(t) = 1 - t$ we get $a_1 = 1$, which gives us the set $\{1\}$ upstairs. From $u(t) = 1$ we get an empty set downstairs. The motivic symbol becomes

$$\frac{\{1\}}{\varnothing}$$

Note 17. The completely multiplicative functions can be characterized by all associated motivic symbols being of the form

1. An empty set or a set containing a single element upstairs, and
2. An empty set downstairs

See Lemmas 3.5.7 and 3.5.8 for precise statements in the case of classical multiplicative functions.

Example 3.2.2. The Möbius function $\mu(n)$ has the Bell series

$$1 - t = \frac{1-t}{1}$$

We have $u(t) = 1 - t$ and $v(t) = 1$. We get an empty set upstairs, and $b_1 = 1$ which gives us the set $\{1\}$ downstairs. The motivic symbol becomes

$$\frac{\varnothing}{\{1\}}$$

Example 3.2.3. The Bell series of the Euler totient function varies with the prime at which we are working at, but can be represented in the table below. It has the general Bell series

$$\begin{aligned} & 1 + (p^1 - p^0)t + (p^2 - p^1)t^2 + (p^3 - p^2)t^3 + \cdots \\ &= (1 + p^1 t + p^2 t^2 + p^3 t^3 + \cdots) - (p^0 t + p^1 t^2 + p^2 t^3 + \cdots) \\ &= \frac{1}{1-pt} - \frac{t}{1-pt} \\ &= \frac{1-t}{1-pt} \end{aligned}$$

We find the Dirichlet series of the function by through the following computation:

$$\prod_p f_p(p^{-s}) =$$

$$\prod_p \frac{1-p^{-s}}{1-p \cdot p^{-s}} =$$

$$\frac{\prod_p (1-p^{-s})}{\prod_p (1-p^{-(s-1)})} =$$

$$\frac{\prod_p \frac{1}{(1-p^{-(s-1)})}}{\prod_p \frac{1}{(1-p^{-s})}} =$$

$$\frac{\zeta(s-1)}{\zeta(s)}$$

In general for the Euler totient function, we have $u(t) = 1 - t$ and $v(t) = 1 - pt$. This gives us $a_1 = p$ and $b_1 = 1$, and the motivic symbol

$$\frac{\{p\}}{\{1\}}$$

	Bell coefficients						Bell series		Motivic symbol	
	e=0	1	2	3	4	5	Num.	Den.	Upst.	Downst.
p=2	1	1	2	4	8	16	$1-t$	$1-2t$	$\{2\}$	$\{1\}$
p=3	1	2	6	18	54	162	$1-t$	$1-3t$	$\{3\}$	$\{1\}$
p=5	1	4	20	100	500	2500	$1-t$	$1-5t$	$\{5\}$	$\{1\}$
p=7	1	6	42	294	2058	14406	$1-t$	$1-7t$	$\{7\}$	$\{1\}$

3.3 More Examples

In this section we provide some more examples of multiplicative functions, their definitions, Dirichlet series, Bell series and motivic symbols. We present the relation between Bell series and motivic symbols in a table. When no other reference is given, the Dirichlet series is either computed by us or taken from the paper "Survey of Dirichlet Series of Multiplicative Arithmetic Functions" by Richard J. Mathar [4].

Example 3.3.1. The indicator function of square numbers is defined as

$$1_{A^2}(n) := \begin{cases} 1 & \text{if } n = m^2 \text{ for some number } m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

The Dirichlet series of $1_{A^2}(n)$ will be

$$1_{A^2}(n) \to \sum_{n=1}^{\infty} \frac{1_{A^2}(n)}{n^s}$$

.

Which can be expressed in terms of the Riemann zeta function as

$$1_{A^2}(n) \to \zeta(2s)$$

The motivic symbol of 1_{A^2} at all primes will be

$$\frac{\{1, -1\}}{\varnothing}$$

	Bell coefficients						Bell series		Motivic symbol	
	e=0	1	2	3	4	5	Num.	Den.	Upst.	Downst.
p=2	1	0	1	0	1	0	1	$1-t^2$	{1, -1}	$\varnothing$
p=3	1	0	1	0	1	0	1	$1-t^2$	{1, -1}	$\varnothing$
p=5	1	0	1	0	1	0	1	$1-t^2$	{1, -1}	$\varnothing$

Example 3.3.2. The indicator function of cube numbers is

$$1_{A^3}(n) := \begin{cases} 1 & \text{if } n = m^3 \text{ for some number } m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

The Dirichlet series of $1_{A^3}(n)$ will be

$$1_{A^3}(n) \to \sum_{n=1}^{\infty} \frac{1_{A^3}(n)}{n^s},$$

and expressed in terms of the Riemann zeta function we have

$$1_{A^3}(n) \to \zeta(3s)$$

Let $\omega = e^{\frac{2i\pi}{3}}$.

The motivic symbol of 1_{A^3} at all primes will be

$$\frac{\{1, \omega, \omega^2\}}{\varnothing}$$

	Bell coefficients						Bell series		Motivic symbol	
	e=0	1	2	3	4	5	Num.	Den.	Upst.	Downst.
p=2	1	0	0	1	0	0	1	$1-t^3$	$\{1, \omega, \omega^2\}$	$\varnothing$
p=3	1	0	0	1	0	0	1	$1-t^3$	$\{1, \omega, \omega^2\}$	$\varnothing$
p=5	1	0	0	1	0	0	1	$1-t^3$	$\{1, \omega, \omega^2\}$	$\varnothing$

Example 3.3.3. The indicator function of fourth powers is defined as

$$1_{A^4}(n) := \begin{cases} 1 & \text{if } n = m^4 \text{ for some number } m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

The Dirichlet series of $1_{A^4}(n)$ will be

$$1_{A^4}(n) \to \sum_{n=1}^{\infty} \frac{1_{A^4}(n)}{n^s},$$

and expressed in terms of the Riemann zeta function we have

$$1_{A^4}(n) \to \zeta(4s)$$

The motivic symbol of 1_{A^4} at all primes will be

$$\frac{\{1, -1, i, -i\}}{\varnothing}$$

	Bell coefficients						Bell series		Motivic symbol	
	e=0	1	2	3	4	5	Num.	Den.	Upst.	Downst.
p=2	1	0	0	1	0	0	1	$1-t^4$	$\{1, -1, i, -i\}$	$\varnothing$
p=3	1	0	0	1	0	0	1	$1-t^4$	$\{1, -1, i, -i\}$	$\varnothing$
p=5	1	0	0	1	0	0	1	$1-t^4$	$\{1, -1, i, -i\}$	$\varnothing$

Example 3.3.4. The indicator function of fifth powers is defined as

$$1_{A^5}(n) := \begin{cases} 1 & \text{if } n = m^5 \text{ for some number } m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

The Dirichlet series of $1_{A^5}(n)$ will be

$$1_{A^5}(n) \to \sum_{n=1}^{\infty} \frac{1_{A^5}(n)}{n^s},$$

and expressed in terms of the Riemann zeta function we have

$$1_{A^5}(n) \to \zeta(5s)$$

Let $\omega = e^{\frac{i\pi}{5}}$.

The motivic symbol of 1_{A^5} at all primes will be

$$\frac{\{1, \omega, \omega^2, \omega^3, \omega^4\}}{\varnothing}$$

	Bell coefficients						Bell series		Motivic symbol	
	e=0	1	2	3	4	5	Num.	Den.	Upst.	Downst.
p=2	1	0	0	0	0	1	1	$1-t^4$	$\{1,\omega,\omega^2,\omega^3,\omega^4\}$	$\varnothing$
p=3	1	0	0	0	0	1	1	$1-t^4$	$\{1,\omega,\omega^2,\omega^3,\omega^4\}$	$\varnothing$
p=5	1	0	0	0	0	1	1	$1-t^4$	$\{1,\omega,\omega^2,\omega^3,\omega^4\}$	$\varnothing$

Example 3.3.5. Let f be the indicator function of square-free numbers. Then the definition of f is

- $f(n) = 1$ if n is a square-free number
- $f(n) = 0$ otherwise

The Dirichlet series will be

$$f(n) \to \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

and in terms of the Riemann zeta function we have

$$f(n) \to \frac{\zeta(s)}{\zeta(2s)}$$

The motivic symbol of f at all primes will be

$$\frac{\varnothing}{\{-1\}}$$

	Bell coefficients						Bell series		Motivic symbol	
	e=0	1	2	3	4	5	Num.	Den.	Upst.	Downst.
p=2	1	1	0	0	0	0	$1+t$	1	$\varnothing$	{-1}
p=3	1	1	0	0	0	0	$1+t$	1	$\varnothing$	{-1}
p=5	1	1	0	0	0	0	$1+t$	1	$\varnothing$	{-1}

Example 3.3.6. The Liouville function, or the unitary Möbius function, $\lambda(n)$ is defined by

$$\lambda(n) = (-1)^{\omega(n)}$$

where $\omega(n)$ is the total number of primes (counted with multiplicity) dividing n.

The Dirichlet series of the function is

$$\lambda(n) \to \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s},$$

The Dirichlet series expressed in terms of the Riemann zeta function is

$$\lambda(n) \to \frac{\zeta(2s)}{\zeta(s)}$$

The motivic symbol of $\lambda(n)$ at all primes will be

$$\frac{\{-1\}}{\varnothing}$$

	Bell coefficients						Bell series		Motivic symbol	
	e=0	1	2	3	4	5	Num.	Den.	Upst.	Downst.
p=2	1	-1	1	-1	1	-1	1	$1+t$	{-1}	$\varnothing$
p=3	1	-1	1	-1	1	-1	1	$1+t$	{-1}	$\varnothing$
p=5	1	-1	1	-1	1	-1	1	$1+t$	{-1}	$\varnothing$

Example 3.3.7. The identity function $Id(n)$ is defined by

$$Id(n) = n$$

for all n.

The function $Id(n)$ has the Dirichlet series

$$Id(n) \to \sum_{n=1}^{\infty} \frac{n}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^{s-1}}$$

and expressed in terms of the Riemann zeta function we get

$$Id(n) \to \zeta(s-1).$$

The general motivic symbol of $Id(n)$ at the prime p will be

$$\frac{\{p\}}{\varnothing}$$

	Bell coefficients						Bell series		Motivic symbol	
	e=0	1	2	3	4	5	Num.	Den.	Upst.	Downst.
p=2	1	2	4	8	16	32	1	$1-2t$	{2}	$\varnothing$
p=3	1	3	9	27	81	243	1	$1-3t$	{3}	$\varnothing$
p=5	1	5	25	125	625	3125	1	$1-5t$	{5}	$\varnothing$

Example 3.3.8. The gamma function $\gamma(n)$ is defined by

$$\gamma(n) = (-1)^{\omega(n)}$$

where $\omega(n)$ is the number of distinct primes dividing n. The function's Dirichlet series is given by

$$\gamma(n) \to \textstyle\prod_p (1-p^{-s})(1-p^{-2s})(1-p^{-3s})^2(1-p^{-4s})^3(1-p^{-5s})^6 \\ \times (1-p^{-s6})^9(1-p^{-7s})^{18}(1-p^{-8s})^{30}\dots$$

Here the exponent of $1-p^{-ms}$ is

$$\frac{1}{m}\left(\sum_{d|n} \mu(\frac{m}{d})(2^d-1)\right)$$

(See OEIS sequence A059966, [5])

The Dirichlet series can not be expressed in terms of $\zeta(s)$.

The motivic symbol of $\gamma(n)$ at all primes will be

$$\frac{\{1\}}{\{2\}}$$

	Bell coefficients						Bell series		Motivic symbol	
	e=0	1	2	3	4	5	Num.	Den.	Upst.	Downst.
p=2	1	-1	-1	-1	-1	-1	$1-2t$	$1-t$	$\{1\}$	$\{2\}$
p=3	1	-1	-1	-1	-1	-1	$1-2t$	$1-t$	$\{1\}$	$\{2\}$
p=5	1	-1	-1	-1	-1	-1	$1-2t$	$1-t$	$\{1\}$	$\{2\}$

Example 3.3.9. The function $\delta(n)$, also called the unit for Dirichlet convolution is defined by

$$\delta(n) := \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

The function has the Dirichlet series

$$\delta(n) \rightarrow \sum_{n=1}^{\infty} \frac{\delta(n)}{n^s},$$

and expressed in terms of the Riemann zeta function we get

$$\delta(n) \rightarrow 1.$$

The motivic symbol of $\delta(n)$ at all primes will be

$$\frac{\varnothing}{\varnothing}$$

	Bell coefficients						Bell series		Motivic symbol	
	e=0	1	2	3	4	5	Num.	Den.	Upst.	Downst.
p=2	1	0	0	0	0	0	1	1	$\varnothing$	$\varnothing$
p=3	1	0	0	0	0	0	1	1	$\varnothing$	$\varnothing$
p=5	1	0	0	0	0	0	1	1	$\varnothing$	$\varnothing$

Example 3.3.10. The function $\sigma_0(n)$ denotes the sum of the 0^{th} power of all the positive divisors of n. In other words, $\sigma_0(n)$ is defined by

$$\sigma_0(n) = d(n)$$

where $d(n)$ is the number of positive divisors of n.

The function will have the Dirichlet series

$$\zeta_D = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}$$

Which can be written as an expression of the ζ-function like

$$\sigma_0(n) \to \zeta^2(s)$$

The motivic symbol of $\sigma_0(n)$ at all primes will be

$$\frac{\{1,1\}}{\varnothing}$$

	Bell coefficients						Bell series		Motivic symbol	
	e=0	1	2	3	4	5	Num.	Den.	Upst.	Downst.
p=2	1	2	3	4	5	6	1	$(1-t)^2$	{1, 1}	$\varnothing$
p=3	1	2	3	4	5	6	1	$(1-t)^2$	{1, 1}	$\varnothing$
p=5	1	2	3	4	5	6	1	$(1-t)^2$	{1, 1}	$\varnothing$

Example 3.3.11. The function $\sigma_1(n)$ is defined by

$$\sigma_1(n) = \sigma(n),$$

where $\sigma(n)$ is the sum of all the positive divisors of n.
The function will have the Dirichlet series

$$\sigma_1(n) \to \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s}$$

Which can be written as an expression of the ζ-function as

$$\sigma_1(n) \to \zeta(s)\zeta(s-1)$$

The general motivic symbol of $\sigma_1(n)$ at the prime p will be

$$\frac{\{1,p\}}{\varnothing}$$

	Bell coefficients						Bell series		Motivic symbol	
	e=0	1	2	3	4	5	Num.	Den.	Upst.	Downst.
p=2	1	3	7	15	31	63	1	$(2t-1)(t-1)$	$\{1,2\}$	$\{\varnothing\}$
p=3	1	4	13	40	121	364	1	$(3t-1)(t-1)$	$\{1,3\}$	$\{\varnothing\}$
p=5	1	6	31	156	781	3906	1	$(5t-1)(t-1)$	$\{1,5\}$	$\{\varnothing\}$

Example 3.3.12. The Möbius function μ is defined by

- $\mu(n) = 1$ if n is a square-free positive integer with an even number of prime factors.
- $\mu(n) = -1$ if n is a square-free positive integer with an odd number of prime factors.
- $\mu(n) = 0$ if n has a squared prime factor.

The Möbius function has the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

which can be written as

$$\mu(n) \to \frac{1}{\zeta(s)}$$

This is the Dirichlet inverse of the Riemann zeta function.

The motivic symbol of $\mu(n)$ at all primes will be

$$\frac{\varnothing}{\{1\}}$$

	Bell coefficients						Bell series		Motivic symbol	
	e=0	1	2	3	4	5	Num.	Den.	Upst.	Downst.
p=2	1	-1	0	0	0	0	$1-t$	1	$\varnothing$	$\{1\}$
p=3	1	-1	0	0	0	0	$1-t$	1	$\varnothing$	$\{1\}$
p=5	1	-1	0	0	0	0	$1-t$	1	$\varnothing$	$\{1\}$

Example 3.3.13. The Jordan's totient function $J_k(n)$ is the number of k-tuples of positive integers all less than or equal to n that form a coprime $(k+1)$-tuple together with n. It can be written as

$$J_k(n) = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right).$$

The Dirichlet series of $J_k(n)$ is

$$J_k(n) \to \sum_{n=1}^{\infty} \frac{J_k(n)}{n^s}$$

The zeta expression is

$$J_k(n) \to \frac{\zeta(s-k)}{\zeta(s)}$$

The motivic symbol at the prime p is

$$\frac{\{p^k\}}{\{1\}}$$

Remark 5. $J_k(n)$is a generalisation of Euler's totient function, which is $J_1(n)$.

Example 3.3.14. The Dedekind psi function $\psi(n)$ is defined by

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right)$$

where the product is taken over all primes p dividing n.

The Dirichlet series of $\psi(n)$ is

$$\psi(n) \to \sum_{n=1}^{\infty} \frac{\psi(n)}{n^s}$$

The zeta expression is

$$\psi(n) \to \frac{\zeta(s)\zeta(s-1)}{\zeta(2s)}$$

The general motivic symbol is

$$\frac{\{p\}}{\{-1\}}$$

Remark 6. The Dedekind psi function can be written as the ratio $J_2(n)/J_1(n)$ where $J_k(n)$ is the Jordan function.

Example 3.3.15. The function $\sigma_k(n)$ is defined as the sum of the k-th powers of the positive divisors of n. It can be expressed in sigma notation as

$$\sigma_k(n) = \sum_{d|n} d^k$$

where $d|n$ is shorthand for "d divides n".

The Dirichlet series of $\sigma_k(n)$ is

$$\sigma_k(n) \to \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s}$$

The zeta expression is

$$\sigma_k(n) \to \zeta(s-k)\zeta(s)$$

The motivic symbol at the prime p is

$$\frac{\{p^k, 1\}}{\varnothing}$$

Example 3.3.16. The function Id_k denotes the power functions, defined by $Id_k(n) = n^k$for any positive integer k.

The Dirichlet series of $Id_k(n)$ is

$$Id_k(n) \to \sum_{n=1}^{\infty} \frac{n^k)}{n^s}$$

The zeta expression is

$$Id_k(n) \to \zeta(s-k)$$

The motivic symbol at the prime p is

$$\frac{\{p^k\}}{\varnothing}$$

Example 3.3.17. Many other multiplicative functions can be obtained by multiplying two multiplicative functions, or by composing them with a power function. Here are some examples of the kind:
The function

$$d(n^2)$$

with zeta expression $\frac{\zeta^3(s)}{\zeta(2s)}$ and motivic symbol $\frac{\{1,1\}}{\{-1\}}$.
The function

$$d(n)^2$$

with zeta expression $\frac{\zeta^4(s)}{\zeta(2s)}$ and motivic symbol $\frac{\{1,1,1\}}{\{-1\}}$.
The function

$$\sigma_k(n) \cdot \sigma_m(n)$$

with zeta expression

$$\frac{\zeta(s)\zeta(s-k)\zeta(s-m)\zeta(s-(k+m))}{\zeta(2s-(k+m))}$$

and motivic symbol

$$\frac{\{1, p^k, p^m, p^{k+m}\}}{\{-p^{\frac{k+m}{2}}, p^{\frac{k+m}{2}}\}}$$

at the prime p.
The function

$$\sigma_k(n^2)$$

with zeta expression

$$\frac{\zeta(s)\zeta(s-k)\zeta(s-2k)}{\zeta(2s-2k)}$$

and motivic symbol

$$\frac{\{1, p^{2k}\}}{\{-p^k\}}$$

at the prime p.

3.4 Multiplicative Functions from Motivic Symbols

In this section we explain how we can associate a multiplicative function to certain motivic symbols. We also define other central terms useful in the statement and proof of the final theorem.

3.4.1 Some Key Definitions

Definition 3.4.1. Let $\mathbb{M}$ denote the monoid $\{\pm P^k \mid k \in \mathbb{N}\}$ under multiplication. Here P is a formal symbol, and $\mathbb{N}$ contains 0.

Remark 7. The monoid $\mathbb{M}$ is isomorphic to $C_2 \times \mathbb{N}$, where C_2 denotes the cyclic group of order 2, defined as the additive structure of $\mathbb{Z}/2$. The monoid $\mathbb{N}$ is defined as the additive structure of the non-negative integers. This isomorphism is given by mapping P^k to $(0, k)$ and $-P^k$ to $(1, k)$.

Definition 3.4.2. Let $\mathbb{L}$ denote the lambda-ring $MS(\mathbb{M})$.

Definition 3.4.3. We define the function **mf** as the map from $\mathbb{L}$ to the set of multiplicative functions such that if the motivic symbol A/B is mapped to the multiplicative function f, then for all primes p,

$$f_p(t) = G(Corr(A/B); t)$$

where we replace the formal symbol P with the prime p at which we are working. We define $Corr$ with the canonical inclusion of $\mathbb{M}$ into the polynomial ring $\mathbb{C}[P]$.

Remark 8. This is well-defined because multiplicative functions are determined by their values at prime powers, and the Bell series contain exactly those values.

Example 3.4.1. When stating that we replace the formal symbol P with for instance the prime 5 in a motivic symbol, what we mean is mapping for instance

$$\frac{\{P, P^2\}}{\{-1\}} \mapsto \frac{\{5, 5^2\}}{\{-1\}} = \frac{\{5, 25\}}{\{-1\}}$$

So, if we for instance want to compute $f := \mathbf{mf}\left(\frac{\{P,P^2\}}{\{-1\}}\right)$, we get the equations

$$f_2(t) = G\left(Corr\left(\frac{\{2,4\}}{\{-1\}}\right); t\right) = \frac{1+t}{(1-2t)(1-4t)}$$

$$f_3(t) = G\left(Corr\left(\frac{\{3,9\}}{\{-1\}}\right); t\right) = \frac{1+t}{(1-3t)(1-9t)}$$

$$f_5(t) = G\left(Corr\left(\frac{\{5,25\}}{\{-1\}}\right); t\right) = \frac{1+t}{(1-5t)(1-25t)}$$

and so on.

Proposition 3.4.1. Let $A/B \in \mathbb{L}$ and let f be a multiplicative function. If (and only if) A/B is the motivic symbol of f at all primes p (we replace the prime p with the formal symbol P), then $\mathbf{mf}(A/B) = f$

Proof. This follows immediately from the definition. □

Definition 3.4.4. Let $A/B \in \mathbb{L}$. We define $\mathbf{ds}(A/B)$ as the Dirichlet series of $\mathbf{mf}(A/B)$.

3.5 The Main Theorem

In this final section we provide an example of an MS-Ring that is relevant in the context of multiplicative functions and Dirichlet series.

3.5.1 Classical Multiplicative Functions

Here, we define the concept of a classical multiplicative function. There are many other ways we have defined this concept, but we could have decided on this one because it includes most of the interesting multiplicative functions one might call "classical", without including too many. In future work we will generalize our results in this paper to many other classes of multiplicative functions.

Definition 3.5.1. A classical multiplicative function is a multiplicative function whose Dirichlet series can be written as a quotient of products of factors of the form $\zeta(t \cdot s - t \cdot k)$, where t is either 1 or 2, and $k \in \mathbb{N}$, and s is the variable.

Example 3.5.1. Of the multiplicative functions in Section 3.3 ("More Examples"), we see that the constant function, the indicator function of square numbers, the indicator function of square-free numbers, the Liouville function, the Euler totient function, the Dirichlet unit δ, σ_k for all non-negative integers k, the Möbius function, Jordan's totient function, Id_k for all positive integers k,

$\sigma_k(n^2)$ for all non-negative integers k, $d(n)^2$ and the Dedekind ψ-function are all classical, while the indicator functions of third, fourth, and fifth powers, as well as the γ-function, are not. The function $\sigma_k(n) \cdot \sigma_m(n)$ is classical if and only if k and m have the same parity.

Theorem 3.5.1. The function **mf** induces a bijection between $\mathbb{L}$ and the set of classical multiplicative functions.

Proof. Let $G(A/B;t)$ be shorthand for $G(Corr(A/B);t)$. First, we need to prove that multiplicative functions from **mf** actually are classical. Let's prove a few lemmas:

Lemma 3.5.2.

$$\mathbf{ds}\left(\frac{A}{B} \oplus \frac{C}{D}\right) = \mathbf{ds}\left(\frac{A}{B}\right) \cdot \mathbf{ds}\left(\frac{C}{D}\right)$$

where $\cdot$ denotes normal product of Dirichlet series (Dirichlet convolution when considering them as multiplicative functions).

Proof. We know from Theorem 2.4.8 that *Corr* maps direct sum to Cauchy product of sequences. By definition and Euler product,

$$\mathbf{ds}\left(\frac{U}{V}\right) = \prod_{p \text{ prime}} G(Corr(U/V); p^{-s})$$

This gives us

$$\mathbf{ds}\left(\frac{A}{B} \oplus \frac{C}{D}\right) = \prod_{p \text{ prime}} G(Corr(A/B); p^{-s}) \cdot G(Corr(C/D); p^{-s})$$

$$= \prod_{p \text{ prime}} G(Corr(A/B); p^{-s}) \cdot \prod_{p \text{ prime}} G(Corr(C/D); p^{-s})$$

$$= \mathbf{ds}\left(\frac{A}{B}\right) \cdot \mathbf{ds}\left(\frac{C}{D}\right)$$

□

Lemma 3.5.3.

$$\mathbf{ds}\left(\ominus\frac{A}{B}\right) = \mathbf{ds}\left(\frac{A}{B}\right)^{-1}$$

where $^{-1}$ denotes the Dirichlet inverse of a Dirichlet series.

Proof. We know from Lemma 3.5.2 that **ds** maps direct sum to product of Dirichlet series. Thus,

$$\mathbf{ds}\left(\frac{A}{B} \oplus \ominus\frac{A}{B}\right) = \mathbf{ds}\left(\frac{A}{B}\right) \cdot \mathbf{ds}\left(\ominus\frac{A}{B}\right)$$

We also know that $\frac{A}{B} \ominus \frac{A}{B} = \frac{\varnothing}{\varnothing}$. Therefore, we have

$$\mathbf{ds}\left(\frac{A}{B}\right) \cdot \mathbf{ds}\left(\ominus\frac{A}{B}\right) = \mathbf{ds}\left(\frac{\varnothing}{\varnothing}\right) = 1$$

This gives us

$$\mathbf{ds}a\left(\ominus\frac{A}{B}\right) = \mathbf{ds}\left(\frac{A}{B}\right)^{-1}$$

□

Note that any element in an MS-Ring can be written as a direct sum of the basis elements and their additive inverses. In $\mathbb{L}$, the basis elements are $\{\pm P^k\}/\varnothing$ for all integers $k \geq 0$. Thus, if we prove that those motivic symbols map to a quotient of products of factors of the form $\zeta(s-a)$ or $\zeta(2s-2a)$ for some non-negative integer a, we have proven that the image of $\mathbf{ds}$ consists only of classical multiplicative functions.

Lemma 3.5.4.

$$\mathbf{ds}\left(\frac{\{P^a\}}{\varnothing}\right) = \zeta(s-a)$$

Proof. This can be shown using the definitions and the Euler product:

$$\zeta(s-a) = \sum_{n=1}^{\infty}\frac{1}{n^{s-a}} = \prod_{p \text{ prime}}\frac{1}{1-p^{-(s-a)}} = \prod_{p \text{ prime}}\frac{1}{1-p^a p^{-s}}$$

Converting this back into a power series of a multiplicative function, we get the Bell series at p:

$$f_p(t) = \frac{1}{1-p^a t}$$

Note that the proof would be completed by showing that

$$G\left(\left(\frac{\{P^a\}}{\varnothing}\right);t\right) = \frac{1}{1-p^a t}$$

In other words,

$$G\left(\left(\frac{\{P^a\}}{\varnothing}\right);t\right)\cdot(1-p^a t) = 1$$

which follows straight from the definition of corresponding sequences.

□

Lemma 3.5.5.

$$\mathbf{ds}\left(\frac{\{-P^a, P^a\}}{\varnothing}\right) = \zeta(2s-2a)$$

Proof. This can also be shown using the definitions and Euler product:

$$\zeta(2s-2a) = \sum_{n=1}^{\infty}\frac{1}{n^{2s-2a}} = \prod_{p \text{ prime}}\frac{1}{1-p^{-(2s-2a)}} = \prod_{p \text{ prime}}\frac{1}{1-(p^a p^{-s})^2}$$

Converting this back into a power series of a multiplicative function (substituting p^s with t), we get the Bell series at p:

$$f_p(t) = \frac{1}{1-(p^a t)^2} = \frac{1}{(1-p^a t)(1+p^a t)}$$

Note that the proof would be completed by showing that

$$G\left(\left(\frac{\{-P^a, P^a\}}{\varnothing}\right); t\right) = \frac{1}{(1-p^a t)(1-(-p^a)t)}$$

In other words,

$$G\left(\left(\frac{\{-P^a, P^a\}}{\varnothing}\right); t\right) \cdot (1-p^a t)(1-(-p^a)t) = 1$$

which also follows straight from the definition of corresponding sequences.

□

Lemma 3.5.4 shows that $\{P^k\}/\varnothing$ maps to $\zeta(s-k)$ for all integers $k \geq 0$, which certainty is a quotient of products of terms of the form $\zeta(s-k)$ and $\zeta(2s-2k)$. $\{-P^k\}/\varnothing$ for all integers $k \geq 0$ requires a tiny bit more work however. Lemma 3.5.5 shows that $\{-P^k, P^k\}/\varnothing$ maps to $\zeta(2s-2k)$ for all k. Note however that $\{-P^k\}/\varnothing = \{-P^k, P^k\}/\varnothing \ominus \{P^k\}/\varnothing$. By Lemmas 3.5.3, 3.5.4 and 3.5.5, this gives

$$\mathbf{ds}\left(\frac{\{-P^k\}}{\varnothing}\right) = \mathbf{ds}\left(\frac{\{-P^k, P^k\}}{\varnothing} \ominus \frac{\{P^k\}}{\varnothing}\right) = \frac{\zeta(2s-2k)}{\zeta(s-k)}$$

for all k. By this, we have that the image of **ds** consists only Dirichlet series of classical multiplicative functions, which again implies that the image of **mf** consists only of classical multiplicative functions.

Now, let's prove the bijectivity of **mf**. Let's start with surjectivity of **ds**. Surjectivity is defined as all Dirichlet series of classical multiplicative functions being mapped to by **ds**. In Lemma 3.5.4 and Lemma 3.5.5, we constructed motivic symbols mapping to $\zeta(s-a)$ and $\zeta(2s-2a)$. All classical multiplicative functions have Dirichlet series that are quotients of products of such Dirichlet series, by definition. A quotient of products is mapped to by direct "subtraction" of two direct sums. Thus, we have a motivic symbol for all Dirichlet series of classical multiplicative functions. This clearly also means that **mf** maps to all classical multiplicative functions.

Now for injectivity. Injectivity of **mf** means that classical multiplicative function are mapped to by at most one motivic symbol. Theorem 2.4.13 gives us that $Corr$ is injective, since the canonical inclusion of the monoid $\mathbb{M}$ into $\mathbb{C}[P]$, which is an integral domain, doesn't map anything to 0. In other words, $Corr(A/B) = Corr(C/D) \Rightarrow A/B = C/D$. Let's try to prove injectivity of **mf** by assuming that two classical multiplicative functions are equal, and show that this implies that their motivic symbols are equal.

Assume $\mathbf{mf}(A/B) = \mathbf{mf}(C/D)$. It is sufficient to show that $Corr(A/B) = Corr(C/D)$. This implies that at all prime powers p^e, $\mathbf{mf}(A/B)(p^e) = \mathbf{mf}(C/D)(p^e)$. Recall that we define $\mathbf{mf}(A/B)(p^e) = Corr(A/B)_e$ (we define this indirectly with Bell series). Thus, we have $Corr(A/B)_n = Corr(C/D)_n$ for all n, which is again equivalent to $Corr(A/B) = Corr(C/D)$. With this, we have proven that both **ds** and **mf** are bijective. □

3.5.2 Isomorphism with $\mathbb{L}$ & Properties

Theorem 3.5.6. The set of all classical multiplicative functions can be equipped with a lambda-ring structure in which

1. Addition corresponds to Dirichlet convolution, and additive inverse corresponds to Dirichlet inverse.
2. Multiplication corresponds to the multiplication of multiplicative functions, provided at least one of the factors is completely multiplicative.
3. The second Adams operation ψ^2 corresponds to taking the norm of multiplicative functions.
4. Furthermore, ψ^2, ψ^4, ψ^8, ψ^{16} and so on correspond to the higher norm operators of multiplicative functions. (see Note 15 for definition)

This lambda-ring is isomorphic to the lambda-ring $\mathbb{L}$.

Remark 9. Given this theorem, a large number of classical identities between multiplicative functions become trivial. For example, the Dirichlet convolution identity

$$\sum_{d|n} J_k(d) \cdot \sigma_0\left(\frac{n}{d}\right) = \sigma_k(n)$$

which in the language of motivic symbols simply becomes

$$\frac{\{p^k\}}{\{1\}} \oplus \frac{\{1,1\}}{\varnothing} = \frac{\{p^k,1\}}{\varnothing}$$

which is obvious. Another example is

$$\sum_{d|n} d^k(J_r)(d) \cdot J_k\left(\frac{n}{d}\right) = J_{k+r}$$

Note that this can be rewritten as

$$\sum_{d|n} (Id_k J_r)(d) \cdot J_k\left(\frac{n}{d}\right) = J_{k+r}$$

Since Id_k is completely multiplicative, we get in the language of motivic symbols:

$$\frac{\{p^k\}}{\varnothing} \otimes \frac{\{p^r\}}{\{1\}} \oplus \frac{\{p^k\}}{\{1\}} = \frac{\{p^{k+r}\}}{\{1\}}$$

From the definition of tensor product, $\frac{\{p^k\}}{\varnothing} \otimes \frac{\{p^r\}}{\{1\}}$ becomes $\frac{\{p^{k+r}\}}{\{p^k\}}$. Now the identity is obvious.

Proof of the Main Theorem. By Theorem 3.5.1, we have a bijection between $\mathbb{L}$ and the set of classical multiplicative functions, and furthermore their Dirichlet series.

Let's prove our statements.

1. Addition corresponds to Dirichlet convolution, and additive inverse corresponds to Dirichlet inverse.

   See Lemma 3.5.2 and Lemma 3.5.3 respectively.

2. Multiplication corresponds to the multiplication of multiplicative functions, provided at least one of the factors is completely multiplicative.

   Lemma 3.5.7. Completely multiplicative functions have motivic symbols of the form $\{a\}/\varnothing$ or $\varnothing/\varnothing$.

   Proof. Let A/B be the motivic symbol corresponding with our completely multiplicative function. In a completely multiplicative function, one will have $f(p^e) = f(p)^e$. This means, by definition, that $G(Corr(A/B);t) = 1 + zt + z^2t^2 + z^3t^3 \dots$ for some complex number z. Note that in Example 2.4.1, we showed that the corresponding sequence of $\{a\}/\varnothing$ is $1, a, a^2, a^3, \dots$ This gives us $G(Corr(\{z\}/\varnothing);t) = 1 + zt + z^2t^2 + z^3t^3 \dots$, which together with $G(Corr(A/B);t) = 1 + zt + z^2t^2 + z^3t^3 \dots$ gives $G(Corr(A/B);t) = G(Corr(\{z\}/\varnothing);t)$, which again is equivalent to $Corr(A/B) = Corr(\{z\}/\varnothing)$. If $z \neq 0$, then $Corr$ is injective on $\mathbb{L}$, so $A/B = \{z\}/\varnothing$. In the case of $z = 0$, we have $A/B = \varnothing/\varnothing$ or $A/B = \{0\}/\varnothing$. The latter is not in $\mathbb{L}$, so we will thus finally have $A/B = \{z\}/\varnothing$ for some $z \in \mathbb{M}$ or $A/B = \varnothing/\varnothing$. ☐

   We know by Theorem 2.4.9 that

   $$Corr\left(\frac{\{a\}}{\varnothing} \otimes \frac{C}{D}\right)_n = Corr\left(\frac{C}{D}\right)_n \cdot a^n$$

   Note that $Corr(\{a\}/\varnothing)$ is $1, a, a^2, a^3, \dots$ Thus, the corresponding sequence of the tensor product $\frac{\{a\}}{\varnothing} \otimes \frac{C}{D}$ is the pointwise product of the corresponding sequences. Thus, what remains to prove is that product of multiplicative functions is equivalent to pointwise product of their Bell series.

   Let f, g be two multiplicative functions. We can create another multiplicative function $f \cdot g$ by multiplying the values of the two multiplicative functions. We can also create the function r which is defined as the pointwise product of the multiplicative functions' Bell series. We want to show that these two are equal for all f, g. Multiplicative functions are determined by their values at prime powers, so it suffices to show that $(f \cdot g)(p^e) = r(p^e)$. We know that $(f \cdot g)(n)$ is defined as $f(n)g(n)$. This let's us rewrite the left-hand side as $f(p^e)g(p^e)$. Recall that the right-hand side was defined as pointwise product of bell series, which are given by values at p^e. This gives us $r(p^e) = f(p^e)g(p^e)$. Clearly, the two are equal.

3. The second Adams operation ψ^2 corresponds to taking the norm of multiplicative functions.

We want to prove that $\mathbf{mf}(\psi^2(A/B)) = N(\mathbf{mf}(A/B))$ for all $A/B \in \mathbb{L}$. Rewrite the motivic symbol A/B as a direct sum and difference of motivic symbols of the form $\{z\}/\varnothing$ for elements $z \in \mathbb{M}$. More specifically, rewrite A/B as

$$A/B = \left(\bigoplus_{a\in A} \frac{\{a\}}{\varnothing}\right) \ominus \left(\bigoplus_{b\in B} \frac{\{b\}}{\varnothing}\right)$$

Putting this into our statement we get:

$$\mathbf{mf}\left(\psi^2\left(\left(\bigoplus_{a\in A} \frac{\{a\}}{\varnothing}\right) \ominus \left(\bigoplus_{b\in B} \frac{\{b\}}{\varnothing}\right)\right)\right)$$

$$= N\left(\mathbf{mf}\left(\left(\bigoplus_{a\in A} \frac{\{a\}}{\varnothing}\right) \ominus \left(\bigoplus_{b\in B} \frac{\{b\}}{\varnothing}\right)\right)\right)$$

Using the fact that ψ^2 is a ring endomorphism and element 1 of this list, we can expand this in the following manner:

$$\mathbf{mf}\left(\psi^2\left(\bigoplus_{a\in A} \frac{\{a\}}{\varnothing}\right) \ominus \psi^2\left(\bigoplus_{b\in B} \frac{\{b\}}{\varnothing}\right)\right)$$

$$= N\left(\mathbf{mf}\left(\bigoplus_{a\in A} \frac{\{a\}}{\varnothing}\right) * \mathbf{mf}\left(\bigoplus_{b\in B} \frac{\{b\}}{\varnothing}\right)^{-1}\right)$$

$$\Leftrightarrow \mathbf{mf}\left(\bigoplus_{a\in A} \psi^2\left(\frac{\{a\}}{\varnothing}\right) \ominus \bigoplus_{b\in B} \psi^2\left(\frac{\{b\}}{\varnothing}\right)\right)$$

$$= N\left(\left(\ast_{a\in A}\mathbf{mf}\left(\frac{\{a\}}{\varnothing}\right)\right) * \left(\ast_{b\in B}\mathbf{mf}\left(\frac{\{b\}}{\varnothing}\right)\right)^{-1}\right)$$

Since $\{a\}/\varnothing$ are completely multiplicative for all a (see Lemma 3.5.8), the above can be rewritten using [2], 1.71 and the definition of ψ^2 as

$$\Leftrightarrow \mathbf{mf}\left(\bigoplus_{a\in A} \frac{\{a^2\}}{\varnothing} \ominus \bigoplus_{b\in B} \frac{\{b^2\}}{\varnothing}\right)$$

$$= \left(\ast_{a\in A}\mathbf{mf}\left(\frac{\{a\}}{\varnothing}\right)^2\right) * \left(\ast_{b\in B}\mathbf{mf}\left(\frac{\{b\}}{\varnothing}\right)^2\right)^{-1}$$

In 2 of this list we proved that $\mathbf{mf}(A/B)\cdot\mathbf{mf}(C/D) = \mathbf{mf}((A/B)\otimes(C/D))$ if either A/B or C/D are of the form $\{a\}/\varnothing$. Thus, we have

$$\Leftrightarrow \mathbf{mf}\left(\bigoplus_{a\in A} \frac{\{a^2\}}{\varnothing} \ominus \bigoplus_{b\in B} \frac{\{b^2\}}{\varnothing}\right)$$

$$= \left(\ast_{a\in A}\mathbf{mf}\left(\frac{\{a^2\}}{\varnothing}\right)\right) * \left(\ast_{b\in B}\mathbf{mf}\left(\frac{\{b^2\}}{\varnothing}\right)\right)^{-1}$$

As we have already shown,

$$\mathbf{mf}\left(\bigoplus_{a\in A}\frac{\{a^2\}}{\varnothing}\ominus\bigoplus_{b\in B}\frac{\{b^2\}}{\varnothing}\right)$$

can be expanded using part 1 of this list. Doing this gives us

$$\left(*_{a\in A}\mathbf{mf}\left(\frac{\{a^2\}}{\varnothing}\right)\right)*\left(*_{b\in B}\mathbf{mf}\left(\frac{\{b^2\}}{\varnothing}\right)\right)^{-1}$$

which is clearly equal to the right-hand side.

Lemma 3.5.8. $\mathbf{mf}(\{a\}/\varnothing)$ is completely multiplicative for all $a\in\mathbb{M}$.

Proof. Let $f=\mathbf{mf}(\{a\}/\varnothing)$. Then f is defined by

$$f_p(t)=G(Corr(\{a\}/\varnothing);t)$$

for all primes p, where we replace the formal symbol P with p. The above is equivalent to

$$f_p(t)=1+at+a^2t^2+a^3t^3+\dots$$

This clearly implies that f is completely multiplicative. □

4. Furthermore, ψ^2, ψ^4, ψ^8, ψ^{16} and so on correspond to the higher norms of multiplicative functions. (see Note 15 for definition)

   We want to prove that $N^r(\mathbf{mf}(A/B))=\mathbf{mf}(\psi^{2^r}(A/B))$ for all $r\geq 1$. Note that $N^r=N\circ N^{r-1}$ by definition, and $\psi^{2^r}=\psi^2\circ\psi^{2^{r-1}}$, by the axioms of Adams operations. We know that $\mathbf{mf}\circ\psi^2=N\circ\mathbf{mf}$. Because mf is bijective, and by induction, $N^r(\mathbf{mf}(A/B))=\mathbf{mf}(\psi^{2^n}(A/B))$.

With all of this, we have proven everything we set out to prove. □

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