Project description:

Grothendieck rings in arithmetic geometry

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1 Introduction

This project proposal concerns research on problems in the interface between number theory, representation theory, algebraic geometry and homotopy theory. The proposed research can be viewed as a "decategorified" approach to arithmetic geometry. The central idea is to introduce new computational methods for the study of Grothendieck rings appearing in arithmetic geometry (the most important examples being various Grothendieck rings of motives) and to connect this study to questions about zeta functions and other classical problems.

2 The research project

In the following pages I will briefly review the background for the project and key challenges to further progress, before outlining the proposed research, divided into 5 major subprojects.

2.1 Background and status of knowledge

2.1.1 Categories in arithmetic geometry. Arithmetic geometers study a wide variety of objects with direct or indirect connections to classical number-theoretic questions about prime numbers and diophantine equations. The objects of interest are organised into *categories*, meaning that they are not viewed in isolation, but studied as classes equipped with various operations (examples might include notions of "direct sum", "tensor product", and "internal Hom") and with *morphisms*, i.e. maps from one object to another. I will assume here that the notion of a category is familiar, but if this is not the case, one may keep in mind the example of finite-dimensional vector spaces over $\mathbb{R}$. The collection of such vector spaces comes with operations called direct sum and tensor product, familiar from linear algebra. The morphisms from a vector space V to another vector space W are simply the linear maps, and the collection of these maps (from V to W) forms a vector space in itself; this is the internal Hom of V and W.

Since the revolutionary work of Grothendieck and his school in the 60s and 70s, arithmetic geometry has to a large extent been formulated in terms of (1) Objects in the category of *schemes* and (2) Objects in various *Tannakian categories*. In the past 20 years or so (but with ideas going back to Quillen in the 70s), it has become increasingly clear that one also needs (3) Objects of a *homotopical* or *higher-categorical* nature, like stacks or motivic homotopy types. We begin by discussing schemes and Tannakian categories, and will return to the subject of stacks towards the very end of the proposal.

2.1.2 Schemes. The formal definition of a scheme uses the technical language of locally ringed spaces, but for practical purposes one can think of schemes in one of two ways. Firstly, a scheme can be seen simply as a system of polynomial equations. For example, the famous Fermat equation $x^n + y^n = z^n$ defines a scheme. For the purposes of this project description we shall tacitly assume that the coefficients of these polynomials are integers (meaning we look at "schemes of finite type over $Spec(\mathbb{Z})$"). Secondly, a scheme can also be seen as a machine (more precisely a *functor*) that takes a commutative ring as input and gives a set as output. For example, the machine that to a given commutative ring associates its set of invertible elements is a scheme.

To any scheme, one can associate three different types of mathematical entities, and arithmetic geometry can to a large extent be described as the study of these three.

1. For a scheme X (thought of as a system of equations) we have its set of solutions in the ring of integers $\mathbb{Z}$, or in any other ring R of interest. For Fermat's equation, an integer solution in the case $n = 2$ is precisely what's classically known as a *Pythagorean triple*, and for $n \geq 3$, "Fermat's Last Theorem" (proved in the Abel-prize-winning work of Wiles) says that a solution in integers can only occur if each variable takes one of the values -1, 0 or 1.

2. To a scheme, one can also associate invariants called *cohomology groups*. These are vector spaces (or more generally abelian groups or objects in some Tannakian category), which reflect various properties of the underlying scheme. A machine assigning cohomology groups to a scheme is called a *cohomology theory*; examples include de Rham cohomology, etale cohomology, motivic cohomology, and algebraic K-theory.

3. Finally, to any scheme (of finite type over $Spec(\mathbb{Z})$) we can associate a local zeta function at a prime p, as well as a global zeta function (taking into account all the primes). These zeta functions are complex-valued functions of a complex variable, and they express in a compact form various collections of data and mysterious patterns at the heart of number theory, from results going back to Euler and Gauss to modern-day mathematics like the proof of Wiles or relations between number theory and the theory of mirror symmetry in mathematical physics.

Example 1. The equation $y^2 + y = x^3 - x^2$ is an example of a special kind of scheme called an *elliptic curve*. It has three etale cohomology groups, of dimension 1, 2, and 1 respectively. There are simple formulas that from these numbers let us compute that this scheme has 1 connected component, that its dimension is 1, and that its genus ("the number of holes in the associated donut, or Riemann surface") is also 1.

2.1.3 Tannakian objects. The definition of a Tannakian category is technical, but a simple example is the category of finite-dimensional real vector spaces. In general, if we pick a field k (for simplicity of characteristic zero) and a group G, the category of k-vector spaces with a G-action is a Tannakian category, which we denote by $Rep_k(G)$. One caveat here is that for general Tannakian categories, G may not be a group in the elementary sense, but something more complicated, like an affine group scheme.

Examples of Tannakian categories of central interest in arithmetic geometry are categories of Galois representations, categories of automorphic representations, and most importantly for this proposal, categories of *motives*. The ideas underlying the theory of motives go back to Grothendieck, Deligne and Beilinson. The main idea is that the category of motives should be a Tannakian category together with a functor from the category of schemes, with the property that every reasonable cohomology theory factors through this functor in an essentially unique way. Due to difficulties arising in the theory of algebraic cycles, we are not yet in a position to prove that such a Tannakian category exists (although there are partial results and much work going on in this direction), but there is a well-developed weaker theory of triangulated categories of motives.

Just like in the case of schemes, any automorphic representation, any Galois representation, or any motive (in each case satisfying some technical assumptions) gives rise to a local zeta function at any prime p, and also a global zeta function which takes all the primes into account.

2.1.4 Zeta functions and cohomological invariants. As already mentioned, many deep number-theoretic patterns can be encoded in a zeta function, and many of the major open problems in number theory are related to these functions. Here I want to focus on a specific type of problem, namely the subject of *special values*. A global zeta function is a meromorphic function of a complex variable s, so we may consider its Laurent series expansion at any given point in the complex plane. Taking this point to be an integer n, the expansion is of the form $c_k(s-n)^k + c_{k+1}(s-n)^{k+1} + \dots$ where k is the vanishing order of the function at n (so a negative value of k occurs if the function has a pole at $s = n$). The coefficient c_k is called the *leading coefficient*, and the theory of special values seeks to determine formulas for the vanishing order and leading coefficient of zeta functions at integer vaues.

Example 2. Consider the scheme given by the equation $x = 0$. The associated local zeta function at a prime p is $1/(1-p^{-s})$, and the global zeta function is the Riemann zeta function $\zeta(s)$, subject to the famous Riemann hypothesis, one of the seven Millennium Problems. An example of a special value in this case is the equation $\zeta(2) = \frac{\pi^2}{6}$, which can be rewritten as a solution to Euler's Basel problem:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$$

Example 3. Continuing Example 1, we may consider the behaviour at $n = 1$ of the L-function of the elliptic curve E (the L-function is here essentially the same thing as the global zeta function). Here the vanishing order and the leading coefficient are described by the *Conjecture of Birch and Swinnerton-Dyer* (BSD conjecture), another of the Millennium Problems.

In general, for a scheme X and an integer n, it is conjectured that the vanishing order and leading coefficient should be expressed in terms of cohomological invariants of X, i.e. there should be a vast generalisation of the BSD conjecture from elliptic curves and $n = 1$ to all schemes and all values of n. The precise mechanisms are still very far from being understood, and part of the problem is that we do not have a sufficiently good framework for understanding cohomology theories for schemes of finite type over $Spec(\mathbb{Z})$. In simpler situations, *motivic homotopy theory* does provide such a framework for cohomology theories for schemes. In my PhD thesis (see [10]), I used the theory of motives and motivic homotopy theory to construct a new cohomology theory for schemes of finite type over $Spec(\mathbb{Z})$, designed to capture precisely the right generalisation of the BSD conjecture to all schemes. However, although one can formulate conjectures using this theory, the lack of a good theoretical framework makes it very hard to prove anything general beyond reformulations of known results.

2.1.5 Towards new foundations for arithmetic geometry? Even though schemes and motives have played an incredibly powerful role in shaping our current understanding of cohomological invariants and zeta functions, it is well known that they are not sufficient for all purposes, and that it would be highly desirable to have a new foundational language of “generalised schemes”, subsuming scheme theory but allowing also for schemes “compactified at the infinite prime”. The “infinite prime” that one would like to have in such a new framework should be responsible for the fact that the “correct” global zeta function is not simply a product of local zeta functions for all the usual prime numbers, but also contains an extra “Gamma factor”, related to the Hodge theory of the underlying scheme (or motive).

A good illustration of the role the infinite prime is supposed to play is given in the very recent preprint of Flach and Morin [5], in which they describe two new (partly conjectural) cohomology theories called Weil-etale cohomology and Weil-Arakelov cohomology, whose structure should explain (in a much more refined way than the constructions of my thesis) the special values of all (local and global) zeta functions of schemes. This beautiful conjectural framework would to a significant extent solve the problem of generalising the BSD conjecture, but relies formally on a so far non-existing theory of “generalised schemes”.

In the best of all worlds, a new theory of generalised schemes would also extend to a homotopy-theoretic framework similar to motivic homotopy theory, giving a structural understanding of Weil-etale cohomology and other conjectural theories related to zeta functions. To paraphrase Kronecker, such a framework was my *Jugendtraum*, and it is indirectly the motivation for many of the ideas in the research programme I will now describe.

2.2 Approaches, hypotheses and choice of method

2.2.1 Philosophy In the preceeding section I attempted to give a brief introduction to the world of schemes, motives, cohomology, zeta functions and Tannakian categories. When I started teaching in a school four and a half years ago, a very natural question was how to share this beautiful world with the talented and highly motivated students I encountered there, and this prompted me to begin developing the ideas of the current proposal.

Given the immense difficulties in studying categories of motives (and various hoped-for generalisations), I propose a study of their *decategorifications*. Roughly speaking, decategorifying a category means forgetting that there are morphisms between objects, and remembering only isomorphism classes of objects together with operations on objects (such as direct sum or tensor product). For the categories of interest to us here, the most important version of decategorification is the *Grothendieck ring*.

The central theme I would like to develop is that rather than working with categories of schemes or stacks, or with the Tannakian categories mentioned, one can *pass to working directly with the associated Grothendieck rings*. This is obviously not an original idea in itself, but I want to argue that by taking this idea seriously, and by focussing on the *algebraic structure* of Grothendieck rings using *concrete representations*, we can formulate many new research problems of intrinsic interest, while also shedding new light on some aspects of the deep and unsolved problems on zeta functions and cohomology.

A very important point is that while decategorification loses information related to the morphisms, we may also gain new structure such as metrics and extra algebraic operations not visible on the level of the category itself.

2.2.2 Review of lambda-rings The algebraic notion of a lambda-ring plays an important role in many areas of mathematics, including topology, representation theory and algebraic geometry. Lambda-rings are also used as the foundation for one of several proposed candidates for a theory of "generalised schemes" (see Borger [2]). For us, lambda-rings are important because the Grothendieck ring of any Tannakian category carries a lambda-ring structure.

We assume that the notion of a *commutative ring* is familiar. A *lambda-ring* is, informally, a commutative ring R "equipped with all possible symmetric operations". The precise definition of "all possible symmetric operations" is expressed in the notion of a *lambda-structure* on a commutative ring. The general definition of "lambda-structure" is given in terms of an infinite sequence $\lambda^0, \lambda^1, \lambda^2, \ldots$ of functions (not ring homomorphisms!) from R to R, satisfying axioms that are a bit complicated. However, when the ring R is torsion-free, there is a simpler equivalent definition which we give here.

Definition 1. Let R be a torsion-free commutative ring. A lambda-structure on R is an infinite sequence of ring homomorphisms $\psi^1, \psi^2, \ldots$ from R to R satisfying the following axioms:

1. $\psi^1(x) = x$ for all $x \in R$
2. $\psi^m(\psi^n(x)) = \psi^{mn}(x)$ for all m, n and all $x \in R$.
3. $\psi^p(x) \equiv x^p \pmod{pR}$ for all prime numbers p and all $x \in R$.

The last condition means that the difference $\psi^p(x) - x^p$ can be written as a multiple of p, in the ring R. The homomorphisms ψ^m are called *Adams operations*.

2.2.3 Tannakian symbols Many different types of algebraic structures come with a notion of *concrete representation*. By a concrete representation, I mean a way of representing elements by a finite amount of data, storable in a computer, such that all the operations of the algebraic structure can be performed by an algorithm carrying out a finite sequence of steps manipulating the concrete representations of elements. Examples: The integers (represented in terms of their binary expansion), the rational numbers (each represented by a pair of integers), finite groups (elements represented by permutations written in cycle notation), finitely generated rings (elements represented by polynomials), and many examples of associative algebras and Lie algebras (elements represented by matrices).

For lambda-rings, the situation is very different, in that the standard supply of examples of lambda-rings are certain sets of power series, and in such a ring an element is in general represented by an *infinite* amount of data. The lambda-ring operations on power series are also rather complicated. In the search for finite versions of these (initally for purely didactical reasons), two of my students and myself were led to thinking about rational, algebraic and holonomic power series, as well as other ways of representing lambda-rings in a computer. The most useful tool created in this process is a device we call a *Tannakian symbol*, thought of as a lambda-ring analogue of a permutation representation, a polynomial representation, or a matrix representation. What follows is a very brief exposition of this idea, but for complete constructions and many concrete examples, I refer to the recently submitted article [9], the student report [3], and the upcoming preprint [4].

A *multiset* is a set which allows for repeated elements. A *commutative monoid* is a set together with a binary operation that is associative, commutative and has an identity element. A *Tannakian symbol with values in a monoid M* is an ordered pair of finite disjoint multisets with elements taken from M. The collection of all such pairs is denoted by $TS(M)$, and an individual symbol is written as a fraction A/B, where A and B are the multisets. Finally, for a set U, we write $TS(M)^U$ for the set of functions from U to $TS(M)$. The first point of these definitions is that for any commutative monoid M and any set U, the sets $TS(M)$ and $TS(M)^U$ can be equipped with natural lambda-ring structures, and in the case of $TS(M)$ we simply recover the monoid algebra of M. In addition, $TS(M)$ comes with a lambda-ring map to $\mathbb{Z}$, and this map induces a filtration on $TS(M)$ called the *gamma filtration*. The second point is that these algebraic constructions are very easy to work with in computations by hand as well as by computer.

Subproject 1. *Develop the abstract theory of Tannakian symbols and lambda-rings represented by such symbols, as well as a computer implementation of Tannakian symbols in SAGE.*

2.2.4 Algebraic structure theory for Grothendieck rings of Tannakian categories Since all of the categories we are interested in are either Tannakian, or admit at least one cohomology functor to a Tannakian category, it is natural to focus on Grothendieck rings of Tannakian categories. For any Tannakian category $\mathscr{C}$, its associated Grothendieck ring is denoted by $K_0(\mathscr{C})$. As an additive group, it is the free abelian group on isomorphism classes of objects, modulo relations given by all short exact sequences. The commutative ring structure comes from a multiplication induced by the tensor product operation, and the lambda-ring structure is induced by exterior power operations in the category.

It is not true that lambda-rings in general are representable by Tannakian symbols, but I have strong reasons to believe the following conjecture.

Conjecture 1. Let $\mathscr{C}$ be a Tannakian category. Let L be the Grothendieck ring of $\mathscr{C}$, or more generally a sub-lambda-ring or a quotient lambda-ring of this Grothendieck ring.

1. There exists a monoid M and a set U such that L admits an injective lambda-ring homomorphism into $TS(M)^U$.

2. If L is a finitely generated as a lambda-ring, then the set U can be taken to be finite.

3. There exists an efficient and practical recognition algorithm associated to L, which takes as input a general element of $TS(M)^U$, and determines whether the element comes from L.

Example 4. The only case (as far as I am aware) of general structure theorems for Grothendieck rings of Tannakian categories is the case of (complex) representations of compact connected complex Lie groups. In this case, it is known that the Grothendieck ring is generated *as a ring* by elements in one-to-one correspondence with the nodes of the associated Dynkin diagram, but is generated *as a lambda-ring* by elements in one-to-one correspondence with the *arms* of the Dynkin diagram - in other words the number of generators required is bounded above by 3 if we work with lambda-rings, but unbounded if we only work in the setting of commutative rings. There are also structure theorems characterizing which lambda-rings can occur as representation rings of these Lie groups, as well as a theorem saying that the Lie group itself is determined by the Grothendieck ring together with one extra piece of data. We refer to Osse [12] and Guillot [7] for more details.

This leads us to the second main subproject of this research proposal.

Subproject 2. *Investigate Conjecture 1 for many concrete examples of Tannakian categories, and apply Tannakian symbols to prove results on the algebraic structure of their Grothendieck rings.*

Here I would like to emphasize that I believe new and interesting mathematics will appear not primarily as a result of proving the conjecture (part 1 and 2 might even be easy, and the third part would have to be made precise), but to work out many explicit cases. For example, in the case when $\mathscr{C} = Rep_{\mathbb{C}}(G)$ (complex representations of some finite group G), it is very easy to prove part 1 and 2 of the conjecture, but to work out concrete cases (for example minimal sizes of U together with a recognition algorithm in the case of symmetric groups S_n for large n) would as far as I understand be a major achievement. In fact, it appears to be the case that unlike for Lie groups, very little is known in general about the lambda-ring structure on the representation ring of a finite group. The theorems we hope to prove may turn out to be direct analogues of the Lie group results, but may also take the form of other kinds of algebraic structure theorems, for special cases or general classes of Grothendieck rings. Examples of Tannakian categories that we have in mind for this subproject also include various kinds of Hodge structures, isocrystals, Nori's Tannakian category associated to a quiver representation, and maybe Deligne's category $Rep(S_t)$.

2.2.5 Multiplicative functions We now turn our attention to a seemingly unrelated area of mathematics, namely the multiplicative functions studied in elementary and analytic number theory. Recall that a *multiplicative function* is a function f from the positive integers to the complex numbers such that $f(1) = 1$ and such that whenever m and n are relatively prime, the equation $f(mn) = f(m)f(n)$ is satisfied. We write $Mult(\mathbb{C})$ for the set of these functions.

Many classical functions from elementary number theory are multiplicative, like the Euler totient function and the Liouville function, but one can also construct a multiplicative function from any zeta function with an Euler product, so that any scheme, motive, Galois representation or automorphic representation with such a zeta function gives rise to a multiplicative function as well. The classical functions satisfy a bewildering plethora of identities, some going back to Dirichlet and Ramanujan, and others being more recent. What my students and I have discovered (using the theory of Tannakian symbols) is that essentially all of these identities can be organised into a new framework of lambda-ring structures on the set of multiplicative functions. Briefly, we have discovered two distinct commutative ring structures on $Mult(\mathbb{C})$, each of them can be equipped with two distinct lambda-ring structures, and it seems to be the case that all known identities (except identities for divisor functions related to finite-dimensionality of spaces of modular forms) are special cases of the lambda-ring axioms in these four structures, and that the lambda-ring framework often provides strong generalisations of previously known results.

Let us consider a few typical examples to illustrate these ideas. The Tannakian symbols of interest here live in the lambda-ring $TS(\mathbb{C}^\times)^\mathbb{P}$, where $\mathbb{C}^\times$ is the commutative monoid of non-zero complex numbers, and $\mathbb{P}$ is the set of prime numbers. In other words, we are looking at pairs of multisets which are functions of a prime p. In these examples, we use only one of the four lambda-ring structures.

Example 5. The Euler totient function $\varphi(n)$ counts the number of elements between 1 and n which are coprime to n, and this function corresponds to the Tannakian symbol $\{p\}/\{1\}$.

The function $d(n)$ counts the number of positive divisors of n, and corresponds to the symbol $\{1,1\}/\emptyset$. Here $\emptyset$ is the empty multiset.

The function $\sigma(n)$ is defined as the *sum* of all positive divisors of n, and corresponds to the symbol $\{1,p\}/\emptyset$.

The identity function $f(n) = n$ has symbol $\{p\}/\emptyset$.

The constant function $f(n) = 1$ has the symbol $\{1\}/\emptyset$.

The Liouville function $\lambda(n)$ is defined as

$$\lambda(n) = (-1)^{\Omega(n)}$$

where $\Omega(n)$ is the number of prime factors of n, counted with multiplicity. Its symbol is $\{-1\}/\emptyset$.

Definition 2. Recall that a multiplicative function f is *completely multiplicative* if the equation $f(mn) = f(m)f(n)$ holds for *all* arguments m and n. A function f is called *specially multiplicative* if the identity

$$f(m)f(n) = \sum_{d|(m,n)} f(\frac{mn}{d^2})g(d) \tag{1}$$

holds for some completely multiplicative function g (which is then uniquely determined by f). Here the sum is taken over all positive divisors of the greatest common divisor of m and n.

Definition 3. For any two multiplicative functions f and g, we define their Dirichlet convolution $f * g$ by the formula

$$(f*g)(n) = \sum_{d|n} f(d)g(\frac{n}{d})$$

in which we sum over all positive divisors of n.

It is well-known that the Dirichlet convolution of two multiplicative functions is again multiplicative.

Example 6. In the language of Tannakian symbols, a multiplicative function is completely multiplicative if and only if for each prime, the associated symbol is of the form $A/\emptyset$ with A containing at most one element (counted with multiplicity). Similarly, one can prove that a function is specially multiplicative if and only if for all primes, the associated symbol is of the form $A/\emptyset$ with A containing at most two elements (counted with multiplicity). So to investigate which of the above examples are specially multiplicative, we do not need to manipulate the identity (1), but simply inspect the symbols, and we conclude that among the functions in the list, all functions except the Euler function are specially multiplicative.

Furthermore, one can prove that in the most interesting case where A has two elements, the completely multiplicative function g appearing in the identity (1) is obtained by applying the second lambda-operation to the function f (this is λ^2 in the introduction above, not to be confused with the Liouville function). One can also prove that this lambda-operation sends a symbol of the form $\{a,b\}/\emptyset$ to the symbol $\{ab\}/\emptyset$. By inspection, we see that the function $d(n)$ satisfies (1) when g is the constant function, and that the function $\sigma(n)$ satisfies (1) with g the identity function (in this last case, the identity was first proved by Ramanujan).

Example 7. For any multiplicative function f, the *norm* $N(f)$ of f is another multiplicative function defined by Redmond and Sivaramakrishnan [13] by the formula

$$N(f)(n) = \sum_{d|n^2} f(n^2/d)\lambda(d)f(d)$$

where the sum is taken over all positive divisors of n^2 and λ is the Liouville function. They also define the *higher norm operators* by setting $N^1(f) = N(f)$ and then using the recursion $N^k(f) = N(N^{k-1}(f))$.

One of the main theorems of *op.cit.* is that for any positive integer k and all specially multiplicative functions f and g, the identity

$$N^k(f*g) = N^k(f)*N^k(g)$$

holds. In our lambda-ring language, the operator N^k is the 2^k'th *Adams operation*, and Dirichlet convolution is *addition*, and so the theorem of Redmond and Sivaramakrishnan holds for all multiplicative functions f and g (not just the specially multiplicative ones!) by the fact that all Adams operations are ring homomorphisms.

These examples were chosen to convey the flavour of the lambda-ring approach to multiplicative functions, but they hardly even begin to skim the surface. A fuller treatment will appear soon in a preprint, joint with my two students Espeseth and Vik [4]; for partial results and many more examples, see in the meanwhile their student project [3].

Subproject 3. *Write up initial results and continue the investigation on the algebraic structure of various classes of multiplicative functions.*

2.2.6 Grothendieck rings of motives Let $\mathscr{C}$ be the category of motives (to be precise here, I could choose to work with $DM_{gm}(Spec\ \mathbb{Z})$). Any motive gives a multiplicative function, and this assignment defines a map from $K_0(\mathscr{C})$ to $Mult(\mathbb{C})$, which is a lambda-ring homomorphism with respect to one of the four lambda-ring structures on $Mult(\mathbb{C})$ (in fact the same structure as the one appearing in the previous subsection). It appears to be folklore knowledge among experts that injectivity of this map (analogous to Conjecture 1) is equivalent to the Tate conjecture on algebraic cycles; in any case, the map exists and can be used to study the lambda-ring structure on $K_0(\mathscr{C})$.

Using Tannakian symbols, I can implement all known operations on motives on the level of the Grothendieck ring, including direct sum, tensor product, exterior powers, symmetric powers, Tate twists, weight decomposition, and internal Hom. In addition, the Grothendieck ring naturally carries extra algebraic and metric structures that are not visible in the category of motives itself; some of these structures seem not to have been studied before. Properties of schemes and motives expressed in the language of algebraic geometry can often be transferred to properties of the associated symbol - a simple example is that the L-function of an elliptic curve gives rise to a specially multiplicative function, with properties described in the previous section.

Subproject 4. *Investigate the algebraic structure of many different Grothendieck rings of motives, using the results and methods from Subproject 1, 2, and 3, together with methods from analytic number theory and the general theory of tensor categories.*

The full Grothendieck ring of motives is countable but very complicated. However, by looking at small subrings or quotient rings, we may take small steps towards a better understanding of the entire ring. For example, one could use the Honda-Tate classification of abelian varieties of finite fields to say something about 1-motives over a finite field, or one could use data from the LMFDB project [11] to look at lambda-rings generated by various sets of motives, for example all motives of analytic conductor bounded by some small positive integer N (each such ring is expected to be finitely generated, and the full Grothendieck ring of motives is the nested union of these rings as N goes to infinity).

Grothendieck rings of motives also occur in mathematical physics in connection with perturbative quantum field theory, but explicit computations is then usually done only in the simple case of rings of mixed Tate motives, while our method of Tannakian symbols makes such computations feasible in much greater generality. As an example of an application related to physics, I hope to relate results and conjectures in arithmetic mirror symmetry for Calabi-Yau varieties to the algebraic structure of the Grothendieck ring in which the motives of these varieties live.

There are numerous other types of computational experiments that become possible using these concrete models for Grothendieck rings, together with data generated by the emerging new generation of highly efficient algorithms for computing zeta functions (exemplified by Harvey's recent Annals paper [8]). To pick one last example, the question of rationality of one of Kapranov's motivic zeta functions (known to hold in some cases but to be false in general) translates into whether a certain explicit sequence is linearly recursive in the Grothendieck ring of motives, and it would be straightforward to investigate what kind of more general recursion formulas these sequences may satisfy when they are not linear (they could for instance turn out to be algebraic or holonomic recurrences).

2.2.8 Towards new foundations? Finally, we return to the question of generalised schemes and possible generalisations of motives and stable motivic homotopy theory. This is speculative, but one could certainly try to write down what axioms Grothendieck rings of these conjectural categories must satisfy in order to be compatible with the Beilinson conjectures and the conjectures on Weil-etale cohomology and zeta values. It is reasonable to expect Hodge structures to play a role, and there should be some close relationship between these new Grothendieck rings and the Grothendieck rings of motives already discussed.

A direction in which I am more confident of making progress is the homotopical direction. More precisely, I can associate multiplicative functions to many examples of stacks (and higher Artin stacks in the sense of Toën), and I have a generalisation of Tannakian symbols which at least for some classes of stacks appear to work in a similar way as the usual Tannakian symbols work for Tannakian categories. Stacks are generalisations of schemes, and they can be viewed as functors from rings to *simplicial sets* instead of just sets. The first major application of higher category theory to an outstanding conjecture in arithmetic geometry is the recent proof of Gaitsgory and Lurie of the Tamagawa number conjecture [6]. The stacks appearing here are classifying stacks of G-bundles on curves over finite fields, and I am confident that via computations with the Siegel formula, one can compute generalised Tannakian symbols for these stacks.

Subproject 5. *Investigate properties of Grothendieck rings of stacks using computations with generalised Tannakian symbols.*

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