Arakelov motivic cohomology and zeta values

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(joint work with Jakob Scholbach)

The aim of the talk was to present a construction of a new cohomology theory for arithmetic schemes, which we call Arakelov motivic cohomology. The motivation for constructing this cohomology theory comes from three sources.

Firstly, our main motivation is the recent insight obtained independently and in different forms by Soulé and Scholbach, that a new type of cohomology is needed as a crucial input in the study of special values of zeta functions and L-functions (see [Sou09] and [Sch10]). Some version of this cohomology theory should also play a role in the Weil-etale framework described by Flach in this volume.

Secondly, it is natural to ask if the relation between Chow groups and motivic cohomology has a counterpart in the arithmetic setting, i.e. if the arithmetic Chow groups defined by Gillet-Soulé [GS90] and Burgos [Bur97] have an extension/generalization analogous to motivic cohomology.

Thirdly, an idea which I believe goes back to Beilinson is that the Beilinson regulator from motivic cohomology to Deligne cohomology should be interpreted as a kind of boundary map in a localization sequence for the inclusion of an arithmetic scheme into its Arakelov compactification. In such a long exact sequence, motivic cohomology and Deligne cohomology would be two of three components, and Arakelov motivic cohomology would be the third.

Working with a number field as base scheme, the last two points have been realized to a large extent by the work of Burgos and Feliu [BF09] on higher arithmetic Chow groups. The main advantage of our new construction is that it also works for schemes and motives over arithmetic base schemes such as Spec $\mathbb{Z}$. This is crucial for all applications to special values. We expect to prove comparison theorems over a number field between Arakelov cohomology groups and the higher arithmetic Chow groups of Burgos and Feliu. A consequence of such a comparison should be the transfer of certain properties, including proper push-forwards, to higher arithmetic Chow groups. This is interesting because it is needed for the formulation of higher arithmetic Riemann-Roch theorems, something which we also hope to address in the future.

The main technical problem in our construction, as well as in constructions of higher arithmetic Chow groups, is to find a lift of the Beilinson regulator to some category in which one can consider its cone (or homotopy fiber). This is a difficult problem. Goncharov [Gon05] constructed such a lift to a map between certain complexes, but was not able to prove that the induced map on cohomology groups actually agrees with the Beilinson regulator (however, this was proved later by Burgos and Feliu). The thesis of Feliu [Fel07] contains a different solution to the problem, but the lift obtained is not an actual map but rather a zig-zag of maps between complexes, and this zig-zag can not be constructed over an arithmetic base scheme. The key idea in our construction is that one can work with motivic spectra instead of complexes. Thanks to the recent foundational work of Ayoub, Cisinski, Déglise and Riou ([Ayo07a], [Ayo07b], [CD07], [CD09], [Rio09]), the problem of lifting the regulator becomes much easier in this setting, and this is what we exploit to construct the Arakelov motivic cohomology groups.

We briefly summarize the main points of the construction. First we must construct a ring spectrum which represents (real) Deligne cohomology for smooth varieties over $\operatorname{Spec}\mathbb{Q}$. This can be done using a slight modification of the method used by Cisinski and Déglise for mixed Weil cohomologies [CD07], provided one uses a good choice of Deligne complexes. After proving that the resulting Deligne spectrum is orientable, one can apply a general theorem of Cisinski and Déglise [CD09, Cor. 13.2.15], which produces a map of ring spectra from Riou's Beilinson spectrum to the Deligne spectrum. (The Beilinson spectrum is constructed by Riou via Adams operations on the algebraic K-theory spectrum, and is weakly equivalent to Voevodsky's Eilenberg-MacLane spectrum with rational coefficients, see [Rio09] and [CD09, 13.1.2].) This map will induce the Beilinson regulator on the level of cohomology groups. The rough idea now is to precompose this regulator map with the canonical map from the integral coefficient Eilenberg-MacLane spectrum, and define the Arakelov motivic cohomology spectrum as the homotopy fiber of this composition. Writing $\eta : \operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}$ for the generic point of $\operatorname{Spec}\mathbb{Z}$ this produces a cofiber sequence

$$\widehat{\mathrm{H}} \to \mathrm{H}_{\mathcal{M}} \to \eta_* \mathrm{H}_{\mathcal{D}}$$

in the model category underlying $\mathbf{SH}(\operatorname{Spec}\mathbb{Z})$. Here $\widehat{\mathrm{H}}$ is the Arakelov motivic cohomology spectrum, $\mathrm{H}_{\mathcal{M}}$ is the Eilenberg-MacLane spectrum (with integral coefficients), and $\mathrm{H}_{\mathcal{D}}$ is the Deligne spectrum.

This definition of Arakelov motivic cohomology appears to be the right one for the special value conjectures formulated by Scholbach and by Soulé. However, for the Weil-etale topology framework of Flach, we will need a modified definition, probably using etale motivic cohomology, and Deligne cohomology with integral coefficients instead of real coefficients. This is something we hope to come back to in a future paper.

Many good functoriality properties of Arakelov motivic cohomology follow rather formally from Ayoub's six functors formalism. In joint work with Peter Arndt, we use the recent results of Hornbostel [Hor10] on model structures on algebras over operads to equip the Arakelov motivic cohomology groups with a product structure.

For more details on the construction and on the applications to zeta values, we refer to the preprint [HS10] and other forthcoming papers.

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