Andreas Holmstrom - Statement of Research

For several years, I have dreamt of employing recent advances in homotopy theory to approach some of the classical and incredibly hard “big questions” of number theory. In the past six months or so, I have had some key insights which now give a new approach to some of the deepest unsolved problems in number theory (the Beilinson conjectures), and also sheds new light on the mysteries surrounding the so called “infinite prime”. These results are currently being written up for my thesis, and they open the door to a rich and unexplored area of “homotopical number theory”.

The key concept in all of this is the notion of a *cohomology theory*. Cohomology is an extremely powerful tool in all parts of geometry and topology, and various forms of cohomology have also been applied with great success to problems in number theory, algebra, and theoretical physics. Many of the deepest insights in all of pure mathematics are intimately connected with the discovery and application of new forms of cohomology. For example, a number of Fields medals have been awarded for such discoveries, and three out of the seven one-million dollar Millennium Problems are directly related to cohomology.

So what is cohomology? The simplest answer is that a cohomology theory is a tool for measuring various properties of geometric objects. A geometric object could be almost anything, for example an infinite-dimensional sphere, a model of the Universe, or the set of solutions to the famous equation in Fermat's Last Theorem. One could say that a cohomology theory for a class of geometric objects is a rule, which to each geometric object assigns an object from algebra, in a way such that properties and relations in the geometric world are reflected in the algebraic world. Such a tool allows us to transfer difficult problems in geometry to the language of algebra, where they are often much easier to handle, partly because they can be processed by a computer.

There are hundreds of different cohomology theories, designed for various purposes and often connected to the deepest problems of mathematics, in particular within number theory and algebraic geometry. They have many properties in common, but they also differ in many ways. One of my long-term dreams has been to find languages or frameworks which would allow us to organise and think systematically about all these cohomology theories at once, rather than just using each of them in isolation. It turns out that there are such frameworks, but they come from a part of mathematics which has not traditionally been associated with number theory, namely abstract homotopy theory, which can be thought of as the study of very general notions of “shape”. Some leading experts refer to the emergence of homotopy-theoretical methods as a paradigm shift in mathematics, and I have started to look at how this paradigm shift would affect number theory.

Roughly, the homotopy-theoretic viewpoint on cohomology says that the natural classes of geometric objects traditionally studied in geometry should be enlarged, to include a vast “infinity-category” of “generalised spaces”. After doing this, the cohomology theory itself, which encodes an enormous amount of data, including information about every geometric object in our huge class, *can now be identified with a single such space*. This is a stunning conceptual simplification, which has already lead to some major advances in algebraic geometry, for example the work of Voevodsky which earned him the Fields medal in 2002.

What I am doing in my research is to take the homotopy-theoretical viewpoint on cohomology and apply it to number theory. Traditionally, number theorists look at spaces of solutions to equations with integer coefficients, for example the famous Fermat equation $x^n + y^n = z^n$, and study these solution spaces using various cohomology theories. My aim is to expand the classical realm of such solution spaces to some kind of generalised spaces *over the prime numbers*. There are already suggestions by leading mathematicians for what these generalised spaces should be in the setting of algebraic geometry – this is the subject of motivic homotopy theory. I have used these constructions in my work, but also investigated whether other approaches to generalised spaces could be more suitable for number theory.

Much of the groundbreaking work on cohomology in the past decades has been related to the study of cohomology theories over a particular prime number (more precisely cohomology for varieties over finite fields), and such cohomologies are now fairly well understood. For many number-theoretic applications, including a conceptual approach to the famous Riemann hypothesis, one would like to find various yet-to-be-discovered cohomology theories *over all prime numbers at once*, which is far more difficult. One of my main thesis results is a method for constructing such cohomology theories, by "gluing" together an infinite number of cohomologies, one for every prime number. This becomes possible only after identifying the cohomology theories with generalised spaces, following the homotopy-theoretic philosophy. The main application of this gluing method so far is the construction of a new sought-after cohomology theory called *higher arithmetic Chow groups*, suitable for studying classical number-theoretic objects, while taking into account data from the so called "infinite prime". The infinite prime is a concept which accounts for several mysterious phenomena in number theory, including fundamental properties of the Riemann zeta function. The cohomology I construct also gives a new approach to the Beilinson conjectures, one of the major conjectural panoramas in number theory and algebraic geometry.

To stand at the edge of this new area of "homotopical number theory" is incredibly inspiring! I have a long list of very promising research ideas, related to the Beilinson conjectures, new forms of generalised spaces, old and new cohomology theories, and much more. A Fellowship at Clare would allow me to devote all my energy in the next few years to these questions.